6.241 Dynamic Systems and Control

Lecture 17: Robust Stability

Readings: DDV, Chapters 19, 20

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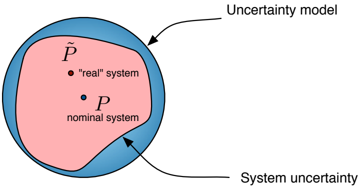
Motivation

- All analytical methods for control design are based on a "model" of the system to be controlled. Such a model does not necessarily represent a perfect description of the system, for several reasons, for example:
  - The complexity of a real physical system can not be handled well by mathematical models.
  - Even in the case in which a perfect model can be designed for a given system, in general the same model is not a perfect description under all operating conditions.
  - The experimental identification of a system's model (including validation of a mathematical model) is very difficult for open-loop unstable plants, for obvious reasons. Even for stable plants, an accurate experimental measurement of high-frequency behavior is very hard to get.
  - Certain characteristics of the model may not be amenable to "easy" control design (e.g., nonlinearities, or very high order or fast dynamics). In these cases, it may be preferable to formulate a simple linear model of the system, that can be used as the basis for control design.

System uncertainty and uncertainty models



- Still, even though a perfect model of the system is not available, it is desired to design an automatic control system that performs according to some specifications not only for the given "model," but also for the "real" system.
- In order to take such uncertainty into account, we will first come up with an uncertainty model, consisting of
  1. A nominal model;
  2. A set of models that is guaranteed to contain the system uncertainty, and is easier to handle.

  and then design a control system that meets the stability and performance specifications not only for P, but also for all other possible models in the uncertainty model.

Multiplicative uncertainty

- Multiplicative uncertainty models are of the form
$$\tilde{P}(s) = (1 + W_2(s)\Delta(s))P(s),$$
in which the frequency weight W_2 is a given stable transfer function, and Δ is an unknown stable transfer function, such that $\|\Delta\|_\infty < 1$.
- Note that since
$$\frac{1}{W_2(s)}\left(\frac{\tilde{P}(s)}{P(s)} - 1\right) = \Delta(s),$$
and $\|\Delta\|_\infty \leq 1$, it must also be true that
$$\left|\frac{1}{W_2(j\omega)}\left(\frac{\tilde{P}(j\omega)}{P(j\omega)} - 1\right)\right| < 1$$
for all frequencies ω, i.e., that
$$\left|\frac{\tilde{P}(j\omega)}{P(j\omega)} - 1\right| < |W_2(j\omega)|, \quad \forall \omega \in \mathbb{R}.$$
- So the multiplicative uncertainty model is a description of how much the ratio of the "real" and "nominal" transfer function is away from being equal to one.

Multiplicative uncertainty

- Among other things, multiplicative uncertainty is useful when the gain of $\tilde{P}$ is uncertain. For example, consider the uncertain system

$$\tilde{P}(s) = \gamma G(s), \quad \gamma \in [\gamma_-, \gamma_+],$$

where $G(s)$ is a known transfer function. We can represent the same set of systems using a multiplicative uncertainty model with

$$P(s) = \gamma_0 G(s), \quad \gamma_0 = \frac{\gamma_- + \gamma_+}{2},$$

and

$$W_2(s) = \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-}.$$

Additive uncertainty

- Additive uncertainty models are of the form

$$\tilde{P}(s) = P(s) + W_2(s)\Delta(s).$$

- Additive uncertainty is useful when the numerator of $\tilde{P}$ contains uncertain terms, such as zero locations.
- For example, consider the case in which

$$\tilde{P}(s) = (s^m + \ldots + \beta s^l + \ldots + 1)G(s), \quad \beta \in [\beta_-, \beta_+].$$

We can represent the same set of systems with an additive uncertainty model of the form

$$P(s) = (s^m + \ldots + \beta_0 s^l + \ldots + 1)G(s), \quad \beta_0 = \frac{\beta_- + \beta_+}{2},$$

and

$$W_2(s) = \frac{\beta_+ - \beta_-}{\beta_+ + \beta_-} s^l.$$

Feedback Uncertainty models

- Feedback uncertainty models are of the form

$$\tilde{P}(s) = \frac{P(s)}{1 + W_2(s)\Delta(s)P(s)}.$$

- Feedback uncertainty is useful when the denominator of $\tilde{P}$ contains uncertain terms, such as damping coefficients.
- For example, consider the case in which

$$\tilde{P}(s) = \frac{G(s)}{s^n + \ldots + \alpha s^l + \ldots + 1}, \quad \alpha \in [\alpha_-, \alpha_+].$$

- We can represent the same set of systems with a feedback uncertainty model of the form

$$P(s) = \frac{G(s)}{s^n + \ldots + \alpha_0 s^l + \ldots + 1}, \quad \alpha_0 = \frac{\alpha_- + \alpha_+}{2},$$

and

$$W_2(s) = \frac{\alpha_+ - \alpha_-}{\alpha_+ + \alpha_-} s^l.$$

Nyquist stability criterion

- Let Γ be a closed path in $\mathbb{C}$, which does not go through any of the zeroes/poles of a function $F : \mathbb{C} \to \mathbb{C}$. The image of Γ under F will encircle the origin $N = Z - P$ times, where Z and P are, respectively, the numbers of zeroes and poles of F inside the contour Γ.
- Consider $F(s) = 1 + L(s)$. The poles of F are the poles of the loop transfer function L; the zeroes of F are the closed-loop poles (i.e., poles of $(1 + L)^{-1}$).
- If Γ is the "D"-contour, traversed clockwise, then the $F \circ \Gamma$ contour will encircle the origin clockwise $N = Z - P$ times.
- In other words, the closed loop will be stable if and only if $P = Z - N$, i.e., if the number of times $L \circ \Gamma$ encircles the -1 point is exactly equal to the number of open-loop unstable poles.

Robust stability

- In the case of additive uncertainty, $L = (P_0 + W\Delta)K = L_0 + W\Delta K$, the Nyquist plot will not encircle the -1 point if $|W(j\omega)K(j\omega)| < |1 + L_0(j\omega)$, i.e., if

$$\left|\frac{W(j\omega)K(j\omega)}{1 + L_0(j\omega)}\right| < 1, \quad \forall \omega \in \mathbb{R}.$$

- Another way to look at this is by this equivalence:

$$\min_{|\Delta(j\omega)| \leq 1} \left|1 + \frac{W(j\omega)K(j\omega)}{1 + P_0(j\omega)K(j\omega)}\Delta(j\omega)\right| > 0, \quad \forall \omega \in \mathbb{R}$$

$$\left|\frac{W(j\omega)K(j\omega)}{1 + L_0(j\omega)}\right| < 1, \quad \forall \omega \in \mathbb{R}.$$

Proof

- $2 \to 1$:

$$\left|1 + \frac{W(j\omega)K(j\omega)}{1 + P_0(j\omega)K(j\omega)}\Delta(j\omega)\right| \geq 1 - \left|\frac{W(j\omega)K(j\omega)}{1 + P_0(j\omega)K(j\omega)}\Delta(j\omega)\right| \geq$$
$$1 - \left|\frac{W(j\omega)K(j\omega)}{1 + P_0(j\omega)K(j\omega)}\right|.$$

- $1 \to 2$: Assume there exists ω_0 for which the ratio is ≥ 1.
- Define

$$\frac{W(j\omega_0)K_{j\omega_0}}{1 + P_0(j\omega_0)K(j\omega_0)} = ae^{j\phi},$$

  if $\Delta(j\omega_0) = \frac{1}{a}e^{-j\phi - j\pi}$, the system is unstable.
- This can be achieved by choosing

$$\Delta(s) = \pm\frac{1}{a}\frac{s - \alpha}{s + \alpha},$$

  since one can always find α such that $\pm\frac{j\omega_0 - \alpha}{j\omega_0 + \alpha} = e^{-j\phi - j\pi}$.

Linear Fractional Description

- Rewrite the block diagram in the following way, i.e., isolating Δ
  [standard $G - \Delta$ block diagram]
- The transfer function of the non-Δ part of the block diagram can be written as

$$\begin{bmatrix} u \\ z \end{bmatrix} = \begin{bmatrix} G_{uy} & G_{uw} \\ G_{zy} & G_{zw} \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix},$$

  where u and y are, respectively, the input and output to Δ.
- The system is assumed to be nominally internally stable (a stabilizing controller may be embedded in G), hence all blocks in G are stable.
- If the interconnection is closed as a feedback on Δ, how do the stability properties of the system change?
- Given $y = \Delta u$, we can solve for the closed-loop transfer function $w \to z$:

$$\Delta u = y = \Delta G_{uy} y + \Delta G_{uw} w \Rightarrow y = (I - \Delta G_{uy})^{-1} \Delta G_{uw} w$$

  and hence

$$z = G_{zy} y + G_{zw} w = \left((I - \Delta G_{uy})^{-1} \Delta G_{uw} + G_{zw}\right) w.$$

"Unstructured" Small-Gain Theorem

- Let $M = G_{uy}$ be the transfer function describing G "as seen by Δ." The robust stability problem is hence equivalent to establishing the stability of

$$(I - \Delta M)^{-1}\Delta = \Delta(I - M\Delta)^{-1},$$

for any stable Δ with $\|\Delta\|_{\mathcal{H}_\infty} < 1$.

Theorem ("Unstructured" Small-Gain Theorem)

Let $\Gamma = \{\Delta \text{ stable } : \|\Delta\|_{\mathcal{H}_\infty} < 1$. If M is stable, then $(I - M\Delta)^{-1}$ and $\Delta(I - M\Delta)^{-1}$ are stable for all $\Delta \in \Gamma$ if and only if $\|M\|_{\mathcal{H}_\infty} < 1$.

Proof — sufficiency

- To prove sufficiency, it is enough to show that if $\|M\|_{\mathcal{H}_\infty} < 1$, $(I - M\Delta)$ has no zeros in the closed RHP.
- For any $x \neq 0$, and s_+ in the closed RHP,

$$\begin{aligned}
\|[I - M(s_+)\Delta(s_+)]x\|_2 &\geq \|x\|_2 - \|M(s_+)\Delta(s_+)x\|_2 \\
&\geq \|x\|_2 - \sigma_{\max}[M(s_+)\Delta(s_+)]\|x\|_2 \\
&\geq \|x\|_2 - \|M\|_{\mathcal{H}_\infty} \|\Delta\|_{\mathcal{H}_\infty}\|x\|_2 > 0
\end{aligned}$$

Proof — necessity

- We want to prove that if $\sigma_{\max}(M(j\omega_0)) > 1$ for some ω_0, then one can construct a Δ with $\|\Delta\|_{\mathcal{H}_\infty} < 1$ that makes the closed-loop system unstable.
- Consider the singular-value decomposition $M(j\omega_0) = U\Sigma V'$, with $\sigma_1 > 1$.

  Then one can choose $\Delta(j\omega_0) = V \begin{bmatrix} 1/\sigma_1 & & \\ & 0 & \\ & & \ddots \end{bmatrix} U'$.
- Clearly,

$$I - M(j\omega_0)\Delta(j\omega_0) = I - U\Sigma V' V \begin{bmatrix} 1/\sigma_1 & & \\ & 0 & \\ & & \ddots \end{bmatrix} U'$$

$$= U \left(I - \begin{bmatrix} 1/\sigma_1 & & \\ & 0 & \\ & & \ddots \end{bmatrix} \right) U' = U \begin{bmatrix} 0 & & \\ & 1 & \\ & & \ddots \end{bmatrix} U',$$

  i.e., a singular matrix.
- It remains to find a transfer function Δ such that $\Delta(j\omega_0)$ has the desired form.

Performance as Stability Robustness

- A performance criterion, e.g., disturbance rejection, can be stated as a bound on the $\mathcal{H}_\infty$ norm of a certain transfer function.

- This is formally the same as the robust stability problem. (i.e., take the "certain transfer function as M)