6.241 Dynamic Systems and Control

Lecture 16: Bode's Sensitivity Integral

Readings: DDV, Chapter 18

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Cauchy's integral theorem

- Let $\Omega \subset \mathbb{C}$ be an open, simply connected set.
- Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. In other words, the limit

$$f'(s_0) = \lim_{s \to s_0} \frac{f(s) - f(s_0)}{s - s_0}$$

  exists (and is continuous) for all $s_0 \in \Omega$. Note that a complex function is holomorphic if and only if it is analytic.
- Let $\gamma : [0, 1] \to \Omega$ be a differentiable function, such that $\gamma(0) = \gamma(1)$.
- Then,

$$\int_0^1 f(\gamma(t))\gamma'(t)\, dt = \int_\Gamma f(z)\, dz = 0,$$

  where Γ is the closed contour traced by $\gamma(t)$ as t ranges from 0 to 1.

Cauchy's integral formula

- Let Ω, f, γ, and Γ be as in the previous slide.
- Furthermore, Let p be a point that is encircled counter-clockwise exactly once by $\gamma(t)$ as t ranges from 0 to 1.
- Then,

$$\int_\Gamma \frac{f(z)}{z-p}\, dz = 2\pi j\, f(p).$$

- Proof sketch:
  - Using Cauchy's integral theorem, show that the value of the integral does not depend on the enclosing path. In particular, we can choose a circle C_ϵ of radius ϵ around p.
  - Note, for $\gamma(t) = p + \epsilon e^{2\pi j t}$, $\displaystyle\int_{C_\epsilon} \frac{1}{z-p}\, dz = 2\pi j \epsilon \int_0^1 \frac{e^{2\pi j t}}{\epsilon e^{2\pi j t}}\, dt = 2\pi j.$
  - Then,

$$\left| \int_{C_\epsilon} \frac{f(z)}{z-p}\, dz - 2\pi j\, f(p) \right| \leq \int_{C_\epsilon} \frac{|f(z)-f(p)|}{|z-p|}\, |dz| \leq$$

$$\int_0^{2\pi} \frac{\max_{z \in C_\epsilon} |f(z)-f(p)|}{\epsilon} \epsilon\, dt \to 0.$$

Bode's Sensitivity Integral for open-loop stable systems

- Let $L(s)$ be a proper, scalar rational transfer function, of relative degree n_r, *i.e., n_r is the difference between the degrees of the denominator and of the numerator.*

- Define $G(s) = (1 + L(s))^{-1}$, and assume that G has neither poles nor zeros in the closed right half plane (i.e., both L and G are stable).

- Then,
$$\int_0^\infty \log |G(j\omega)| \, d\omega = \begin{cases} 0 & \text{if } n_r > 1, \\ -\kappa\frac{\pi}{2} & \text{if } n_r = 1, \end{cases}$$
where $\kappa = \lim_{s\to\infty} sL(s)$.

Proof sketch

- Since $\log|G(s)|$ is analytic on the RHP, then $\int_D \log|G(s)|\, ds = \int_{C_i} \log|G(s)|\, ds + \int_{C_\infty} \log|G(s)|\, ds = 0$, i.e.,

$$j\int_0^\infty \log|G(j\omega)|\, d\omega = \frac{1}{2}\int_{C_\infty} \log|1+L(s)|\, ds.$$

- For large s, $\log|1+L(s)| \approx \log|1+as^{-n_r}| \approx |as^{-n_r}|$, so on C_E,

$$\frac{1}{2}\int_{C_E} \log|1+L(s)|\, ds \approx -\int_0^{\pi/2} \left|\frac{a}{E^{n_r}} e^{-jn_r t}\right| \cdot Eje^{jt}\, dt =$$

$$-\frac{aj}{E^{n_r-1}}\int_0^{\pi/2} e^{jt}\, dt = -\frac{aj}{E^{n_r-1}}\frac{\pi}{2}$$

- In other words, for $n_r > 1$, the integral on C_E converges to 0, and for $n_r = 1$, it converges to $\kappa\frac{\pi}{2}j$. □

Bode's Sensitivity Integral

- Let $L(s)$ be a proper, scalar rational transfer function, of relative degree n_r.
- Define $G(s) = (1 + L(s))^{-1}$, and assume that G has no poles in the closed right half plane (i.e., G is stable), and has $q \geq 0$ zeros in the closed RHP plane (i.e., L can be unstable), at location $z_1, z_2, \ldots, z_q$, with $\mathrm{Re}(z_i) \geq 0$.
- Then,

$$\int_0^\infty \log |G(j\omega)| \, d\omega = \begin{cases} \pi \sum_{i=1}^q z_i & \text{if } n_r > 1, \\ -\kappa \frac{\pi}{2} + \pi \sum_{i=1}^q z_i & \text{if } n_r = 1, \end{cases}$$

where $\kappa = \lim_{s \to \infty} sL(s)$.

Proof sketch

- The function $\log|G(s)|$ is not analytic on the RHP. Define $\hat{G}(s) = G(s)\prod_{i=1}^{q} \frac{s+z_i}{s-z_i}$.
- Since $\log|\hat{G}(s)|$ is analytic on the RHP, then $\int_D \log|\hat{G}(s)|\, ds = 0$, and hence

$$\int_{C_i} \log|G(s)|\, ds + \int_{C_\infty} \log|G(s)|\, ds + \sum_{i=1}^{q} \int_D \log\left|\frac{s+z_i}{s-z_i}\right| ds = 0.$$

- Proceeding as in the basic case, and noting that

$$\int_{C_i} \log\left|\frac{s+z_i}{s-z_i}\right| ds = 0,$$

and

$$\int_{C_\infty} \log\left|\frac{s+z_i}{s-z_i}\right| ds = \int_{C_\infty} \log\left|1+\frac{z_i}{s}\right| ds - \int_{C_\infty} \log\left|1-\frac{z_i}{s}\right| ds = -j\pi z_i,$$

- We get the desired result. Notice also that $\sum_i \mathrm{Re}(z_i) = \sum_i z_i$. □

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