6.241 Dynamic Systems and Control

Lecture 13: I/O Stability

Readings: DDV, Chapters 15, 16

Emilio Frazzoli

Aeronautics and Astronautics
Massachusetts Institute of Technology

March 14, 2011

$\mathcal{L}_2$-induced norm

Theorem ($\mathcal{H}_\infty$ norm is the $\mathcal{L}_2$-induced norm)

The $\mathcal{L}_2$-induced norm of a causal, CT, LTI, stable system S with impulse response $H(t)$ and transfer function $H(s)$ is

$$\|S\|_{2,\mathrm{ind}} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[H(j\omega)] = \|H\|_\infty.$$

- From Parseval's equality, $\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y'(j\omega)Y(j\omega)\,d\omega$.
- Hence,

$$\|y\|_2^2 \leq \frac{1}{2\pi}\int_{\mathbb{R}} \sigma_{\max}(H(j\omega))^2 U'(j\omega)U(j\omega)\,d\omega \leq \sup_{\omega} \sigma_{\max}[H(j\omega)]^2 \|u\|_2^2.$$

- To show the bound is tight, pick (SISO case) $u(t) = exp(\epsilon t + j\omega_0 t)$, i.e., $U(s) = 1/(s - \epsilon - j\omega_0)$, with $\epsilon < 0$. Then, $\|y\|_2^2 = |H(\epsilon + j\omega_0)|^2 \|u\|_2^2$
- As $\epsilon \to 0$, by the continuity of H on the imaginary axis, the gain approaches $|H(j\omega_0)|$.

Computation of $\mathcal{H}_\infty$ norm

Theorem

Let $H(s) = C(sI - A)^{-1}B$ be the transfer function of a stable, strictly causal ($D = 0$) LTI system. Define

$$M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -\frac{1}{\gamma}C^TC & -A^T \end{bmatrix}.$$

Then $\|H\|_\infty < \gamma$ if and only if M_γ has no purely imaginary eigenvalues.

- This allows using bisections to compute $\|H\|_\infty$ to arbitrary precision.
- Similar formulas exist for the general case ($D \neq 0$), but are more complicated.

Computation of $\mathcal{H}_\infty$ norm

[diagram with $H(s)$ and $H^T(-s)$ in unit positive feedback]

- $\|H\| < \gamma$ if and only if $I - \frac{1}{\gamma^2}H'(j\omega)H(j\omega)$ is invertible for all $\omega \in \mathbb{R}$, i.e., if and only if $G_\gamma(s) = \left[I - \frac{1}{\gamma^2}H^T(-s)H(s)\right]^{-1}$ has no poles on the imaginary axis.
- The next step is to build a realization of $G_\gamma(s)$.
- $H^T(-s) = -B^T(sI + A)^{-T}C^T$, so a realization of this is $(-A^T, -C^T, B^T, 0)$.
- Putting together the realizations, and eliminating the internal variables, one gets the system matrix of the realization we seek as

$$M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -C^TC & -A^T \end{bmatrix}. \qquad \square$$

Energy of the impulse response

- Consider a stable, strictly causal CT LTI system with state-space model $(A, B, C, 0)$.
- The energy of the response to an unit impulse can be computed as

$$\|H\|^2_{\mathcal{L}_2} = \mathrm{Tr}\left[\int_0^{+\infty} H(t)^T H(t)\ dt\right] = \frac{1}{2\pi}\mathrm{Tr}\left[\int_{-\infty}^{+\infty} H(j\omega)' H(j\omega)\ ds\right] = \|H\|^2_{\mathcal{H}_2},$$

- This can be computed exactly noting that

$$\|H\|^2_{\mathcal{L}_2} = \mathrm{Tr}\left[\int_0^{+\infty} Ce^{At}BB^Te^{A^Tt}C^T\ dt\right] = \mathrm{Tr}\left[CPC^T\right],$$

where P (called the controllability gramian) can be computed through the Lyapunov equation

$$AP + PA^T + BB^T = 0.$$

Some remarks on the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms[1]

- There is no general relationship between $\mathcal{H}_\infty$ and $\mathcal{H}_\infty$.
- For example, consider

$$G_1(s) = \frac{1}{\epsilon s + 1} \qquad G_2(s) = \frac{\epsilon s}{s^2 + \epsilon s + 1}$$

As $\epsilon \to 0$, $\|G_1\|_\infty = \|G_2\|_\infty = 1$, but $\|G_1\|_2 \to \infty$, and $\|G_2\|_2 \to 0$.

- The $\mathcal{H}_\infty$ norm is an induced norm; then, the sub-multiplicative property holds, i.e., for any $G_1, G_2 \in \mathcal{H}_\infty$,

$$\|G_1 G_2\|_{\mathcal{H}_\infty} \leq \|G_1\|_{\mathcal{H}_\infty} \|G_2\|_{\mathcal{H}_\infty}$$

- The $\mathcal{H}_2$ norm is not an induced norm. So, in general, the submultiplicative property does not hold.

[1] John Wen, 2006

6.241J / 16.338J Dynamic Systems and Control
Spring 2011