6.241 Dynamic Systems and Control

Lecture 12: I/O Stability

Readings: DDV, Chapters 15, 16

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March 14, 2011

Introduction

- Last week, we looked at notions of stability for state-space systems, with no inputs.
- Now we want to consider notions of stability under the effect of a (forcing) input.
- Central to the discussion is the notion of norm of a signal—which is just the same we already discussed, when considering signals as infinite-dimensional vectors.
- In the following, let $w : \mathbb{T} \to \mathbb{R}^n$, with $w(t) = \begin{bmatrix} w_1(t) & w_2(t) & \dots & w_n(t) \end{bmatrix}$.

Signal norms

∞-norm: Peak magnitude

$$\|w\|_\infty = \sup_{t\in\mathbb{T}} \|w(t)\|_\infty = \sup_{t\in\mathbb{T}} \max_{i=1\ldots n} |w_i(t)|$$

2-norm: (Square root of the) Energy

$$\|w\|_2^2 = \begin{cases} \sum_{k\in\mathbb{Z}} w[k]'w[k] = \sum_{k\in\mathbb{Z}} \|w[k]\|_2^2 & \text{(DT)} \\ \int_{-\infty}^{\infty} w(t)'w(t)\, dt = \int_{-\infty}^{\infty} \|w(t)\|_2^2\, dt & \text{(CT)} \end{cases}$$

Power (NOT a norm!)

$$P_w = \rho_w^2 = \begin{cases} \lim_{N\to+\infty} \frac{1}{2N} \sum_{k=-N}^{N} w[k]'w[k] = \lim_{N\to+\infty} \frac{1}{2N} \sum_{k=-N}^{N} \|w[k]\|_2^2 & \text{(DT)} \\ \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} w(t)'w(t)\, dt = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|w(t)\|_2^2\, dt & \text{(CT)} \end{cases}$$

1-norm: Action

$$\|w\|_1 = \begin{cases} \sum_{k\in\mathbb{Z}} \|w[k]\|_1 & \text{(DT)} \\ \int_{-\infty}^{\infty} \|w(t)\|_1\, dt & \text{(CT)} \end{cases}$$

Some examples

- Let $w(t) = \bar{w}, \forall t \in \mathbb{T}$. Then,
  - $\|w\|_\infty = |\bar{w}|$;
  - $\|w\|_2 = +\infty$;
  - $\rho_w = |\bar{w}|$;
  - $\|w\|_1 = +\infty$.
- Let $w(t) = \bar{w}e^{-at}, \forall t \in \mathbb{R}_{\geq 0}$, and $a > 0$. Then,
  - $\|w\|_\infty = |\bar{w}|$;
  - $\|w\|_2 = |\bar{w}|/\sqrt{2a}$;
  - $\rho_w = 0$;
  - $\|w\|_1 = |\bar{w}|/a$.

System Norms

- Recall that a I/O model of a system is an operator mapping an input signal u to an output signal y, i.e., $y = Su$.
- We can define an induced norm for a system in exactly the same way as we did for matrices, i.e.,
$$\|S\|_{p,\mathrm{ind}} := \sup_{u \neq 0} \frac{\|Su\|_p}{\|u\|_p}$$
- We will see how to compute system norms later.

Input-Output stability

Definition (Input-Output stability)

A system with I/O model S is p-stable (or ℓ_p-stable, or $\mathcal{L}_p$-stable), if and only if its p-induced norm is finite, i.e., $\|S\|_{p,\mathrm{ind}} < \infty$. In particular, a system is Bounded-Input, Bounded-Output stable if and only if it is ∞-stable.

- Example: an integrator is not BIBO stable, and not p-stable for any p.

BIBO stability of CT LTI systems

- The I/O model of a LTI system with m inputs and p outputs can be described by an impulse response matrix, $H : \mathbb{T} \to \mathbb{R}^{p \times m}$, whose elements $h_{ij} : \mathbb{T} \to \mathbb{R}$ represent the impulse response from input j to output i.

$$y_i(t) = \int_{-\infty}^{\infty} h_{ij}(t-\tau) u_j(\tau) \, d\tau.$$

Theorem

A CT LTI system S with impulse response matrix H is BIBO stable if and only if

$$\|S\|_{\infty,\text{ind}} = \max_{1 \le i \le p} \sum_{j=1}^{m} \int_{-\infty}^{+\infty} |h_{ij}(t)| \, dt < \infty.$$

- Note: in the scalar case (SISO), $\|S\|_{\infty,\text{ind}} = \|h\|_1$, i.e., the ∞-induced norm of the system S is the $\mathcal{L}_1$ norm of the impulse response h (seen as a signal in the time domain). Often $\|S\|_{\infty,\text{ind}}$ is referred to as the $\mathcal{L}_1$ norm of H in the general, MIMO case as well.

Proof

- Sufficiency:

$$\|y\|_\infty = \sup_{t\in\mathbb{R}} \max_{1\le i\le p} \left| \sum_{j=1}^{m} \int_{-\infty}^{+\infty} h_{ij}(t-\tau)u_j(\tau)\, d\tau \right|$$

$$\le \sup_{t\in\mathbb{R}} \max_{1\le i\le p} \sum_{j=1}^{m} \int_{-\infty}^{+\infty} |h_{ij}(t-\tau)|\, d\tau \cdot \|u\|_\infty$$

$$\le \|S\|_{\infty,\mathrm{ind}} \, \|u\|_\infty$$

- Necessity:
  - Focus on the scalar case, i.e., $\int_{\mathbb{R}} |h(t)| dt = \infty$.
  - Choose u such that $u(t) = -\mathrm{sign}(h(-t))$. Clearly, $\|u\|_\infty \le 1$.
  - Then $y(0) = \int_{\mathbb{R}} h(0-\tau)u(\tau)\, d\tau = \int_{\mathbb{R}} |h(\tau)|\, d\tau = \infty$

Additional remarks

- A similar result holds in discrete time.
- For finite-dimensional LTI systems, one can construct a state-space model, and compute
$$H(t) = Ce^{At}B + D\delta(t), \qquad t \geq 0,$$
which has Laplace transform
$$H(s) = C(sI - A)^{-1}B + D.$$
The system is BIBO stable if and only if the poles of $H(s)$ are in the open left half plane.
- Asymptotic stability implies BIBO stability, but not viceversa.
- For LTI systems, BIBO stability implies p-stability for any p.
- For time-varying and nonlinear systems, the statements above do not necessarily hold.

$\mathcal{L}_2$-induced norm

Theorem ($\mathcal{H}_\infty$ norm is the $\mathcal{L}_2$-induced norm)

The $\mathcal{L}_2$-induced norm of a causal, CT, LTI, stable system S with impulse response $H(t)$ and transfer function $H(s)$ is

$$\|S\|_{2,\mathrm{ind}} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[H(j\omega)] = \|H\|_\infty .$$

- From Parseval's equality, $\|y\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y'(j\omega) Y(j\omega)\, d\omega$.
- Hence,

$$\|y\|_2^2 \leq \frac{1}{2\pi} \int_{\mathbb{R}} \sigma_{\max}(H(j\omega))^2 U'(j\omega) U(j\omega)\, d\omega \leq \sup_{\omega} \sigma_{\max}[H(j\omega)]^2 \|u\|_2^2 .$$

- To show the bound is tight, pick (SISO case) $u(t) = exp(\epsilon t + j\omega_0 t)$, i.e., $U(s) = 1/(s - \epsilon - j\omega_0)$, with $\epsilon < 0$. Then, $\|y\|_2^2 = |H(\epsilon + j\omega_0)|^2 \|u\|_2^2$
- As $\epsilon \to 0$, by the continuity of H on the imaginary axis, the gain approaches $|H(j\omega_0)|$.

Computation of $\mathcal{H}_\infty$ norm

Theorem

Let $H(s) = C(sI - A)^{-1}B$ be the transfer function of a stable, strictly causal $(D = 0)$ LTI system. Define

$$M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -\frac{1}{\gamma}C^TC & -A^T \end{bmatrix}.$$

Then $\|H\|_\infty < \gamma$ if and only if M_γ has no purely imaginary eigenvalues.

- $\|H\| < \gamma$ if and only if $I - \frac{1}{\gamma^2}H'(j\omega)H(j\omega)$ is invertible for all $\omega \in \mathbb{R}$, i.e., if and only if $G_\gamma(s) = \left[I - \frac{1}{\gamma^2}H^T(-s)H(s)\right]^{-1}$ has no poles on the imaginary axis.
- The next step is to build a realization of $G_\gamma(s)$.

Computation of $\mathcal{H}_\infty$ norm

diagram with $H(s)$ and $H^T(-s)$ in unit positive feedback

- $H^T(-s) = -B^T(sI + A)^{-T}C^T$, so a realization of this is $(-A^T, -C^T, B^T, 0)$.
- Putting together the realizations, and eliminating the internal variables, one gets the system matrix of the realization we seek as

$$M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -C^TC & -A^T \end{bmatrix},$$

which proves the claim.

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Spring 2011