6.241 Dynamic Systems and Control

Lecture 9: Transfer Functions

Readings: DDV, Chapters 10, 11, 12

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Asymptotic Stability (Preview)

- We have seen that the unforced state response ($u = 0$) of a LTI system is easily computed using the "A" matrix in the state-space model:

$$x[k] = A^k x[0], \quad \text{or} \quad x(t) = e^{At} x(0).$$

- A system is asymptotically stable if $\lim_{t \to +\infty} x(t) = 0$, for all x_0.
- Assume A is diagonalizable, i.e., $V^{-1}AV = \Lambda$, and let $r = Vx$ be the vector of model coordinates. Then,

$$r_i[k] = \lambda_i^k r_i[0], \quad \text{or} \quad r_i(t) = e^{\lambda_i t} r_i(0), \quad i = 1, \ldots, n.$$

- Clearly, for the system to be asymptotically stable, $|\lambda_i| < 1$ (DT) or $\mathrm{Re}(\lambda_i) < 0$ (CT) for all $i = 1, \ldots, n$.
- It turns out that this condition extends to the general (non-diagonalizable) case. More on this later in the course.

(Time-domain) Response of LTI systems — summary

- Based on the discussion in previous lectures, the solution of initial value problems (i.e., the response) for LTI systems can be written in the form:

$$y[k] = CA^k x[0] + C \sum_{i=0}^{k-1} \left(A^{k-i-1} B u[i] \right) + Du[t]$$

or

$$y(t) = C \exp(At) x(0) + C \int_0^t \exp(A(t-\tau)) Bu(\tau)\, d\tau + Du(t).$$

- However, the convolution integral (CT) and the sum in the DT equation are hard to interpret, and do not offer much insight.
- In order to gain a better understanding, we will study the response to elementary inputs of a form that is
  - particularly easy to analyze: the output has the same form as the input.
  - very rich and descriptive: most signals/sequences can be written as linear combinations of such inputs.
- Then, using the superposition principle, we will recover the response to general inputs, written as linear combinations of the "easy inputs."

The continuous-time case: elementary inputs

- Let us choose as elementary input $u(t) = u_0 e^{st}$, where $s \in \mathbb{C}$ is a complex number.
- If s is real, then u is a simple exponential.
- If $s = j\omega$ is imaginary, then the elementary input must always be accompanied by the "conjugate," i.e.,

$$u(t) + u^*(t) = u_0 e^{j\omega t} + u_0 e^{-j\omega t} = 2u_0 \cos(\omega t);$$

in other words, if s is imaginary, then $u(t) = e^{st}$ must be understood as a "half" of a sinusoidal signal.

- if $s = \sigma + j\omega$, then

$$u(t) + u^*(t) = u_0(e^{\sigma t} e^{j\omega t} + u_0 e^{\sigma t} e^{-j\omega t})$$
$$= u_0(e^{\sigma t} \left(e^{j\omega t} + e^{-j\omega t}\right)) = 2u_0 e^{\sigma t} \cos(\omega t),$$

and the input u is a "half" of a sinusoid with exponentially-changing amplitude.

Output response to elementary inputs (1/2)

- Recall that,

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t).$$

- Plug in $u(t) = u_0 e^{st}$:

$$\begin{aligned} y(t) &= Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu_0 e^{s\tau}\, d\tau + Du_0 e^{st} \\ &= Ce^{At}x(0) + C\left(\int_0^t e^{(sI-A)\tau}\, d\tau\right) e^{At}Bu_0 + Du_0 e^{st} \end{aligned}$$

- If $(sI - A)$ is invertible (i.e., s is not an eigenvalue of A), then

$$y(t) = Ce^{At}x(0) + C(sI-A)^{-1}\left[e^{(sI-A)t} - I\right] e^{At}Bu_0 + Du_0 e^{st}.$$

Output response to elementary inputs (2/2)

- Rearranging:

$$y(t) = \underbrace{Ce^{At}x(0) - C(sI - A)^{-1}e^{At}Bu_0}_{\text{Transient response}} + \underbrace{\left[C(sI - A)^{-1}B + D\right]u_0 e^{st}}_{\text{Steady-state response}}.$$

- If the system is asymptotically stable, $e^{At} \to 0$, and the transient response will converge to zero.
- The steady state response can be written as:

$$y_{\mathrm{ss}} = G(s)e^{st}, \qquad G(s) \in \mathbb{C}^{n_y \times n_u},$$

  where $G(s) = C(sI - A)^{-1}B + D$ is a complex matrix.
- The function $G : s \to G(s)$ is also known as the transfer function: it describes how the system transforms an input e^{st} into the output $G(s)e^{st}$.

Laplace Transform

- The (one-sided) Laplace transform $F : \mathbb{C} \to \mathbb{C}$ of a sequence $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined as

$$F(s) = \int_0^{+\infty} f(t) e^{-st} \, dt,$$

for all s such that the series converges (region of convergence).

- Given the above definition, and the previous discussion,

$$Y(s) = G(s)U(s).$$

$$U(s)e^{st} \quad \Rightarrow \quad Y(s)e^{st} = G(s)U(s)e^{st}$$

- Also, $G(s)$ is the Laplace transform of the "impulse" response.

The discrete-time case: elementary inputs

- Let us choose as elementary input $u[k] = u_0 z^k$, where $z \in \mathbb{C}$ is a complex number.
- If z is real, then u is a simple geometric sequence.
- Recall

$$y[k] = CA^k x[0] + C \sum_{i=0}^{k-1} A^{k-i-1} B u[i] + D u[k].$$

- Plug in $u[k] = u_0 z^k$, and substitute $l = k - i - 1$:

$$\begin{aligned} y[k] &= CA^k x[0] + C \sum_{l=0}^{k-1} A^l B u_0 z^{k-l-1} + D u_0 z^k \\ &= CA^k x[0] + C z^{k-1} \left(\sum_{i=0}^{k-1} (A z^{-1})^i \right) B u_0 + D u_0 z^k. \end{aligned}$$

Matrix geometric series

- Recall the formula for the sum of a geometric series:

$$\sum_{i=0}^{k-1} m^i = \frac{1-m^k}{1-m}.$$

- For a matrix:

$$\sum_{i=0}^{k-1} M^i = I + M + M^2 + \ldots M^{k-1}.$$

$$\sum_{i=0}^{k-1} M^i(I-M) = (I + M + M^2 + \ldots M^{k-1})(I-M) = I - M^k.$$

i.e.,

$$\sum_{i=0}^{k-1} M^i = (I - M^k)(I-M)^{-1}.$$

Discrete Transfer Function

- Using the result in the previous slide, we get

$$y[k] = CA^k x[0] + Cz^{k-1}(I - A^k z^{-k})(I - Az^{-1})^{-1}Bu_0 + Du_0 z^k$$
$$= CA^k x[0] + C(z^k I - A^k)(zI - A)^{-1}Bu_0 + Du_0 z^k.$$

- Rearranging:

$$y[k] = \underbrace{CA^k \left(x[0] - (zI - A)^{-1}Bu_0\right)}_{\text{Transient response}} + \underbrace{\left(C(zI - A)^{-1}B + D\right) u_0 z^k}_{\text{Steady-state response}}.$$

- If the system is asymptotically stable, the transient response will converge to zero.
- The steady state response can be written as:

$$y_{\mathrm{ss}}[k] = G(z)z^k, \qquad G(z) \in \mathbb{C},$$

where $G(z) = C(zI - A)^{-1}B + D$ is a complex number.
- The function $G : z \to G(z)$ is also known as the (pulse, or discrete) transfer function: it describes how the system transforms an input z^k into the output $G(z)z^k$.

Z-Transform

- The (one-sided) z-transform $F : \mathbb{C} \to \mathbb{C}$ of a sequence $f : \mathbb{N}_0 \to \mathbb{R}$ is defined as

$$F(z) = \sum_{k=0}^{+\infty} f[k] z^{-k},$$

for all z such that the series converges (region of convergence).

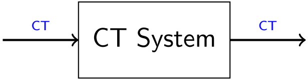
- Given the above definition, and the previous discussion,

$$Y(z) = G(z)U(z).$$

$$U(z)z^k \quad \Rightarrow \quad Y(z)z^k = G(z)U(z)z^k$$
$$Y(z) = G(z)U(z)$$

- Also, $G(z)$ is the z transform of the "impulse" response, i.e., the response to the sequence $u = (1, 0, 0, \ldots)$.

Models of continuous-time systems



$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

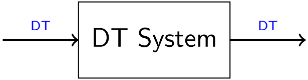
$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & \dots & 1 & 0 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix} \qquad D = d$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = \frac{b_{n-1}s^{n-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} + d$$

Models of discrete-time systems



$$
\begin{aligned}
x[k+1] &= Ax[k] + Bu[k] \\
y[k] &= Cx[k] + Du[k]
\end{aligned}
$$

$$
A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & \dots & 1 & 0 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}
$$

$$
C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix} \qquad D = d
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G(z) = \frac{b_{n-1}z^{n-1} + \dots + b_0}{z^n + a_{n-1}z^{n-1} + \dots + a_0} + d
$$

6.241J / 16.338J Dynamic Systems and Control

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