6.241 Dynamic Systems and Control

Lecture 8: Solutions of State-space Models

Readings: DDV, Chapters 10, 11, 12 (skip the parts on transform methods)

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Forced response and initial-conditions response

- Assume we want to study the output of a system starting at time t_0, knowing the initial state $x(t_0) = x_0$, and the present and future input $u(t)$, $t \geq t_0$. Let us study the following two cases instead:
  - Initial-conditions response:

$$\begin{cases} x_{\mathrm{IC}}(t_0) = x_0, & \\ u_{\mathrm{IC}}(t) = 0, & t \geq t_0, \end{cases} \quad \rightarrow \quad y_{\mathrm{IC}};$$

  - Forced response:

$$\begin{cases} x_{\mathrm{F}}(t_0) = 0, & \\ u_{\mathrm{F}}(t) = u(t), & t \geq t_0, \end{cases} \quad \rightarrow \quad y_{\mathrm{F}}.$$

- Clearly, $x_0 = x_{\mathrm{IC}} + x_{\mathrm{F}}$, and $u = u_{\mathrm{IC}} + u_{\mathrm{F}}$, hence

$$y = y_{\mathrm{IC}} + y_{\mathrm{F}},$$

  that is, we can always compute the output of a linear system by adding the output corresponding to zero input and the original initial conditions, and the output corresponding to a zero initial condition, and the original input.
- In other words, we can study separately the effects of non-zero inputs and of non-zero initial conditions. The "complete" case can be recovered from these two.

Initial-conditions response (DT)

Consider the case of zero input, i.e., $u = 0$; in this case, the state-space equations are written as the difference equations

$$\begin{array}{ll} x[0] = x_0 & y[0] = C[0]x_0 \\ x[1] = A[0]\,x[0] & y[1] = C[1]\,A[0]\,x[0] \\ x[2] = A[1]\,A[0]\,x[0] & y[2] = C[2]\,A[1]\,A[0]\,x[0] \\ \ldots & \ldots \\ x[k] = \Phi[k,0]\,x[0] & y[k] = C[k]\,\Phi[k,0]\,x[0] \end{array}$$

where we defined the state transition matrix $\Phi[k, \ell]$ as

$$\Phi[k,\ell] = \begin{cases} A[k-1]\,A[k-2]\ldots A[l], & k > \ell \geq 0 \\ I, & k = \ell \end{cases}$$

Forced response with zero i.c. (DT)

- We need to compute the solution of $x[k+1] = A_\mathrm{d} x[k] + B_\mathrm{d} u[k]$, $x[0] = 0$.
- By substitution, we get:

$$\begin{aligned} x[k] &= A[k-1]x[k-1] + B[k-1]u[k-1] \\ &= A[k-1](A[k-2]x[k-2] + B[k-1]u[k-2]) + B[k-1]u[k-1] \\ &= \underbrace{\Phi[k,0]\,x[0]}_{=0} + \sum_{i=0}^{k-1} \Phi[k,i+1]B[i]u[i]. \end{aligned}$$

- In other words, $x[k] = \Gamma[k,0]\mathcal{U}[k,0]$, where

$$\Gamma[k,0] = \begin{bmatrix} \Phi[k,1]B[0] & \Phi[k,2]B[1] & \dots & B[k-1] \end{bmatrix}, \qquad \mathcal{U} = \begin{bmatrix} u[0] \\ u[1] \\ \dots \\ u[k-1] \end{bmatrix}.$$

- The output is

$$y[k] = C[k]\Gamma[k,0]\mathcal{U}[k,0].$$

Summary (DT)

- In general, state/output trajectories of a DT state-space model can be computed as:

$$x[k] = \Phi[k,0]x[0] + \Gamma[k,0]\mathcal{U}[k,0],$$

$$y[k] = C[k]\Phi[k,0]x[0] + C[k]\Gamma[k,0]\mathcal{U}[k,0].$$

- In general $\Phi[k,\ell]$ may not be invertible. In the cases in which it is, one can also compute $x[0]$ as a function of $x[k]$.

Initial-conditions response (CT)

- Consider the case of zero input, i.e., $u = 0$; in this case, the state-space equations are written as

$$\begin{aligned} \frac{d}{dt}x(t) &= A(t)x(t), \qquad x(t_0) = x_0; \\ y(t) &= C(t)x(t). \end{aligned}$$

- Assume that the matrix function $A : t \mapsto A(t)$ is sufficiently well behaved so that there exists unique state/output signals x and y. (e.g., A is piecewise-continuous).
- Define a state transition function $\Phi(t, \tau)$ such that, for all $t, \tau \in \mathbb{T}$,

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau),$$

$$\Phi(t, t) = I.$$

- The function Φ can in general be computed numerically, integrating a differential equation in n unknown functions, with n initial conditions (assuming $x \in \mathbb{R}^n$).
- Then, $x(t) = \Phi(t, t_0)x_0$, and $y(t) = C(t)\Phi(t, t_0)x_0$.

Forced response with zero i.c. (CT)

- We need to integrate

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = 0,$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

.

- Again, assume the input signal u and the matrix functions A and B are such that there exists a unique solution.
- Claim: the forced solution is

$$x(t) = \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau.$$

- The output is

$$y = C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(t)\ d\tau + D(t)u(t).$$

- Verify by substitution: clearly $x(t_0) = 0$; moreover,

$$\frac{d}{dt}x(t) = \frac{d}{dt}\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\, d\tau =$$

$$\int_{t_0}^{t} \frac{\partial}{\partial t}\Phi(t,\tau)B(\tau)u(\tau)\, d\tau + [\Phi(t,\tau)B(\tau)u(\tau)]_{\tau=t}$$

$$= A(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\, d\tau + B(t)u(t) = A(t)x(t) + B(t)u(t).$$

- Similarly for the output.

Further properties of the state transition function

- $\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)$.

- Look up on the lecture notes.

The LTI case

- In DT, if $A[k] = A$, $B[k] = B$, for all $k \in \mathbb{T}$, then $\Phi[k, \ell] = A^{k-\ell}$, and $\Gamma[k, \ell] = \begin{bmatrix} A^{k-1}B, & A^{k-2}B, & \dots, & B \end{bmatrix}$.

- in CT, if $A(t) = A$, and $B(t) = B$, for all $k \in \mathbb{T}$, then $\Phi(t, \tau) = \exp(A(t - \tau))$, where

$$\exp(M) := \sum_{i=0}^{+\infty} \frac{1}{i!} M^i = I + M + \frac{1}{2} M^2 + \frac{1}{6} M^3 + \dots$$

- Easy to check that the matrix exponential satisfies the conditions for the state transition function.

Similarity Transformations

- The choice of a state-space model for a given system is not unique.
- For example, let T be an invertible matrix, and set $x = Tr$, i.e., $r = T^{-1}x$. This is called a similarity transformation.
- The standard state-space model can be written as

$$\begin{aligned} Tr^+ &= ATr + Bu \\ y &= CTr + Du \end{aligned}$$

i.e.,

$$\begin{aligned} r^+ &= (T^{-1}AT)r + (T^{-1}B)u = \hat{A}r + \hat{B}u \\ y &= (CT)r + Du = \hat{C}r + \hat{D}u \end{aligned}$$

Modal Coordinates

- Is a state trajectory of the form $x[k] = \lambda^k v$ ($\lambda \neq 0$) a valid solution of the state-space model, assuming $u = 0$?
- Since $x[k+1] = Ax[k]$, then $\lambda^{k+1} v = A\lambda^k v$, i.e., $(\lambda I - A)v = 0$: the proposed state trajectory is a valid solution if and only if v is (right) eigenvector of A, with eigenvalue λ. It will in fact be a solution of the system with initial condition $x[0] = v_i$.
- Assume that A has n independent eigenvectors. Then, any initial condition can be written uniquely as a linear combination of eigenvectors, i.e., $x[0] = \sum_{i=1}^{n} \alpha_i v_i$. The solution of the state-space model is then

$$x[k] = \sum_{i=1}^{n} \alpha_i v_i \lambda_i^k,$$

which is called the modal decomposition of the unforced response.

Modal contributions

- Since $\alpha = V^{-1}x(0)$, one can also write

$$x[k] = \sum_{i=1}^{n} \lambda_i^k v_i w_i' x_0,$$

which shows that $\alpha_i = w_i' x_0$ is the contribution of the initial condition to the i-th mode.

Diagonalization of the system

- If $T = V =$ matrix of eigenvectors, then $V^{-1}AV = \Lambda$ (prove by $AV = V\Lambda$).
- Decoupled system for each mode.