6.241 Dynamic Systems and Control

Lecture 7: State-space Models

Readings: DDV, Chapters 7,8

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February 25, 2011

Outline

1. State-space models

State of a system

We know that, if a system is causal, in order to compute its output at a given time t_0, we need to know "only" the input signal over $(-\infty, t_0]$. (Similarly for DT systems.)

This is a lot of information. Can we summarize it with something more manageable?

Definition (state)

The state $x(t_1)$ of a causal system at time t_1 is the information needed, together with the input u between times t_1 and t_2, to uniquely predict the output at time t_2, for all $t_2 \geq t_1$.

In other words, the state of the system at a given time summarizes the whole history of the past inputs $-\infty$, for the purpose of predicting the output at future times.

Usually, the state of a system is a vector in some Euclidean space $\mathbb{R}^n$.

Dimension of a system

The choice of a state for a system is not unique (in fact, there are infinite choices, or realizations).

However, there are come choices of state which are preferable to others; in particular, we can look at "minimal" realizations.

Definition (Dimension of a system)

We define the dimension of a causal system as the minimal number of variables sufficient to describe the system's state (i.e., the dimension of the smallest state vector).

We will deal mostly with finite-dimensional systems, i.e., systems which can be described with a finite number of variables.

Some remarks on infinite-dimensional systems

Even though we will not address infinite-dimensional systems in detail, some examples are very useful:

- (CT) Time-delay systems: Consider the very simple time delay S_T, defined as a continuous-time system such that its input and outputs are related by

$$y(t) = u(t - T).$$

  In order to predict the output at times after t, the knowledge of the input for times in $(t - T, t]$ is necessary.

- PDE-driven systems: Many systems in engineering, arising, e.g., in structural control and flow control applications, can only be described exactly using a continuum of state variables (stress, displacement, pressure, temperature, etc.). These are infinite-dimensional systems.

In order to deal with infinite-dimensional systems, approximate discrete models are often used to reduce the dimension of the state.

State-space model

Finite-dimensional linear systems can always be modeled using a set of differential (or difference) equations as follows:

Definition (Continuous-time State-Space Models)

$$
\begin{aligned}
\frac{d}{dt}x(t) &= A(t)x(t) + B(t)u(t); \\
y(t) &= C(t)x(t) + D(t)u(t);
\end{aligned}
$$

Definition (Discrete-time State-Space Models)

$$
\begin{aligned}
x[k+1] &= A[k]x[k] + B[k]u[k]; \\
y[k] &= C[k]x[k] + D[k]u[k];
\end{aligned}
$$

The matrices appearing in the above formulas are in general functions of time, and have the correct dimensions to make the equations meaningful.

LTI State-space model

If the system is Linear Time-Invariant (LTI), the equations simplify to:

Definition (Continuous-time State-Space Models)

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t); \\ y(t) &= Cx(t) + Du(t); \end{aligned}$$

Definition (Discrete-time State-Space Models)

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k]; \\ y[k] &= Cx[k] + Du[k]; \end{aligned}$$

In the above formulas, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}$, and n is the dimension of the state vector.

Example of DT system: accumulator

- Consider a system such that
$$y[k] = \sum_{i=-\infty}^{k-1} u[i].$$
- Notice that we can rewrite the above as
$$y[k] = \left(\sum_{i=-\infty}^{k-2} u[i] \right) + u[k-1] = y[k-1] + u[k-1].$$
- In other words, we can set $x[k] = y[k]$ as a state, and get the following state-space model:
$$\begin{aligned} x[k+1] &= x[k] + u[k], \\ y[k] &= x[k]. \end{aligned}$$
- Let $x[0] = y[0] = 0$, and $u[k] = 1$; we can solve by repeated substitution:
$$\begin{aligned} x[1] &= x[0] + u[0] = 0 + 1 = 1, \quad y[1] = x[1] = 1; \\ x[2] &= x[1] + u[1] = 1 + 1 = 2, \quad y[2] = x[2] = 2; \\ & \dots \\ x[k] &= x[k-1] + u[k-1] = k - 1 + 1 = k, \quad y[k] = x[k] = k; \end{aligned}$$

Finite-dimensional Linear Systems 1/2

- Recall the definition of a linear system. Essentially, a system is linear if the linear combination of two inputs generates an output that is the linear combination of the outputs generated by the two individual inputs.
- The definition of a state allows us to summarize the past inputs into the state, i.e.,

$$u(t), -\infty \leq t \leq +\infty \qquad \Leftrightarrow \qquad \begin{cases} x(t_0), & \\ u(t), & t \geq t_0, \end{cases}$$

  (similar formulas hold for the DT case.)

- We can extend the definition of linear systems as well to this new notion.

Finite-dimensional Linear Systems 2/2

Definition (Linear system (again))

A system is said a Linear System if, for any $u_1, u_2, t_0, x_{0,1}, x_{0,2}$, and any two real numbers α, β, the following are satisfied:

$$\left\{ \begin{array}{l} x(t_0) = x_{0,1}, \\ u(t) = u_1(t), \quad t \geq t_0, \end{array} \right. \quad \rightarrow \quad y_1,$$

$$\left\{ \begin{array}{l} x(t_0) = x_{0,2}, \\ u(t) = u_2(t), \quad t \geq t_0, \end{array} \right. \quad \rightarrow \quad y_2,$$

$$\left\{ \begin{array}{l} x(t_0) = \alpha x_{0,1} + \beta x_{0,2}, \\ u(t) = \alpha u_1(t) + \beta u_2(t), \quad t \geq t_0, \end{array} \right. \quad \rightarrow \quad \alpha y_1 + \beta y_2.$$

Similar formulas hold for the discrete-time case.

Forced response and initial-conditions response

- Assume we want to study the output of a system starting at time t_0, knowing the initial state $x(t_0) = x_0$, and the present and future input $u(t)$, $t \geq t_0$. Let us study the following two cases instead:
  - Initial-conditions response:

$$\begin{cases} x_{\mathrm{IC}}(t_0) = x_0, \\ u_{\mathrm{IC}}(t) = 0, \quad t \geq t_0, \end{cases} \quad \rightarrow \quad y_{\mathrm{IC}};$$

  - Forced response:

$$\begin{cases} x_{\mathrm{F}}(t_0) = 0, \\ u_{\mathrm{F}}(t) = u(t), \quad t \geq t_0, \end{cases} \quad \rightarrow \quad y_{\mathrm{F}}.$$

- Clearly, $x_0 = x_{\mathrm{IC}} + x_{\mathrm{F}}$, and $u = u_{\mathrm{IC}} + u_{\mathrm{F}}$, hence

$$y = y_{\mathrm{IC}} + y_{\mathrm{F}},$$

  that is, we can always compute the output of a linear system by adding the output corresponding to zero input and the original initial conditions, and the output corresponding to a zero initial condition, and the original input.
- In other words, we can study separately the effects of non-zero inputs and of non-zero initial conditions. The "complete" case can be recovered from these two.

6.241J / 16.338J Dynamic Systems and Control
Spring 2011