6.241 Dynamic Systems and Control

Lecture 6: Dynamical Systems

Readings: DDV, Chapter 6

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February 23, 2011

Signals

- Signals: maps from a set $\mathbb{T}$ to a set $\mathbb{W}$.
  - Time axis $\mathbb{T}$: topological semigroup[1], in practice $\mathbb{T} = \mathbb{Z}, \mathbb{R}, \mathbb{N}_0$, or $\mathbb{R}_{\geq 0}$, and combinations thereof, such as $\mathbb{Z} \times \mathbb{R}$.
  - Signal space $\mathbb{W}$: vector space, typically $\mathbb{R}^n$, for some fixed $n \in \mathbb{N}$.
- Discrete-time signals ℓ: maps from $\mathbb{Z}$ (or $\mathbb{N}_0$) to $\mathbb{R}^n$.
- Continuous-time signals $\mathcal{L}$: maps from $\mathbb{R}$ (or $\mathbb{R}_{\geq 0}$) to $\mathbb{R}^n$. Typically, constraints are imposed on maps to qualify as continuous-time signals:
  - Piecewise-continuity, or
  - Local (square) integrability.
- DT and CT signals can be given the structure of vector spaces in the obvious way (i.e., time-wise addition and scalar multiplication of signal values).
- It is possible to mix DT and CT signals (e.g., to describe digital sensing of physical processes, zero-order holds, etc.).

[1] Semigroup: group without identity and/or inverse.

Outline

1. Behavioral Models
2. Input-Output Models

Behavioral Models 1/2

A system can be defined as a set of constraints on signals:

Behavioral model of a dynamical system

Given a time axis $\mathbb{T}$ and a signal space $\mathbb{W}$, a behavioral model of a system is a subset $\mathbb{B}$ of all possible signals $\{w : \mathbb{T} \to \mathbb{W}\}$.

- A system is linear if its behavioral model is a vector space, i.e., if

$$w_a, w_b \in \mathbb{B} \Rightarrow \alpha w_a + \beta w_b \in \mathbb{B}, \qquad \forall \alpha, \beta \in \mathbb{F}.$$

- A system is time-invariant if its behavioral model is closed with respect to time shift.
  - For any signal $w : \mathbb{T} \to \mathbb{W}$, define the time-shift operator σ_τ as $(\sigma_\tau w)(t) = w(t - \tau)$
  - A system is time-invariant if $w \in \mathbb{B} \Rightarrow \sigma_\tau w \in \mathbb{B}$, for any $\tau \in \mathbb{T}$.

- A system is memoryless if, for any $v, w \in \mathbb{B}$, and any $T \in \mathbb{T}$, the signal $e : \mathbb{T} \to \mathbb{W}$ defined as

$$e(t) = \begin{cases} v(t) & \text{if } t \leq T \\ w(t) & \text{if } t > T \end{cases}$$

  is also in $\mathbb{B}$. In other words, a system is memoryless if possible futures are independent of the past.

- A system is strictly memoryless if there exists a function $\phi : \mathbb{T} \times \mathbb{W} \to \{\texttt{True}, \texttt{False}\}$ such that $w \in \mathbb{B} \Leftrightarrow \phi(t, w(t)) = \texttt{True}$.
  In other words, a system is strictly memoryless if the constraints imposed on the signals are purely algebraic, point-wise in time (e.g., no derivatives, integrals, etc.).

- Note: any notion of regularity imposed on the signals (as a whole), such as piecewise continuity, integrability, etc. requires a system not to be strictly memoryless. (CT systems always have some kind of memory.)

Example: Memoryless vs. Strictly Memoryless systems

- Consider a behavioral model $\mathbb{B}$ such that $w \in \mathbb{B}$ if and only if w is piecewise constant, i.e., if there exists a finite partition of $\mathbb{T}$ into sets over which w is constant.
- This system is memoryless, but is not strictly memoryless.

Outline

1. Behavioral Models
2. Input-Output Models

Input-Output models 1/2

- Behavioral models treat all components of signals constrained by the system equally, without any differences in their role or interpretation.
- In many applications, it is useful to make a distinction between some of the components of the signals (called the input) and the others (called the output).
- An input-output model is a map S from a set of input signals $\{u : \mathbb{T}_{\text{in}} \to \mathbb{W}_{\text{in}}\}$ and a set of output signals $\{y : \mathbb{T}_{\text{out}} \to \mathbb{W}_{\text{out}}\}$.
- In behavioral terms, an input-output model S is the set $\mathbb{B} = \{(u, y) : y = Su\}$.
- Typically we will consider deterministic input-output, i.e., systems that associate a unique output signal to each input signal, where the time axis is $\mathbb{Z}$, $\mathbb{R}$, or combinations thereof.
- For convenience, we will often assume $\mathbb{T}_{\text{in}} = \mathbb{T}_{\text{out}} = \mathbb{T}$.

Input-Output models 2/2

Properties of behavioral models map easily to input-output models.

- An input-output system S is linear if, for all input signals u_a, u_b,

$$S(\alpha u_a + \beta u_b) = \alpha(S u_a) + \beta(S u_b) = \alpha y_a + \beta y_b, \qquad \forall \alpha, \beta \in \mathbb{F}.$$

- An input-output system S is time-invariant if it commutes with the time-shift operator, i.e., if

$$S \sigma_\tau u = \sigma_\tau S u = \sigma_\tau y \qquad \forall \tau \in \mathbb{T}.$$

- An input-output system S is memoryless (or static) if there exists a function $f : W_{\mathrm{in}} \to W_{\mathrm{out}}$ such that, for all $t_0 \in \mathbb{T}$,

$$y(t_0) = (Su)(t_0) = f(u(t_0)).$$

Causality

- An input-output system S is causal if, for any $t \in \mathbb{T}$, the output at time t depends only on the values of the input on $(-\infty, t]$.
- In other words, define the truncation operator P as

$$(P_T u)(t) = \begin{cases} u(t) & \text{for } t \leq T \\ 0 & \text{for } t > T. \end{cases}$$

  Then an input-output system S is causal if

$$P_T S P_T = P_T S, \qquad \forall T \in \mathbb{T}.$$

- An input-output system S is strictly causal if, for any $t \in \mathbb{T}$, the output at time t depends only on the values of the input on $(-\infty, t)$.

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Spring 2011