6.241 Dynamic Systems and Control

Lecture 5: Matrix Perturbations

Readings: DDV, Chapter 5

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Outline

1. Matrix Perturbations

Introduction

- Important issues in engineering, and in systems and control science in particular, concern the sensitivity of computations, solution algorithms, design methods, to uncertainty in the input parameters.

- For example: What is the smallest perturbation (e.g., in terms of 2-norm) that makes a matrix singular? What is the impact on the solution of a least-square problem of uncertainty in the data? etc.

Additive Perturbation

Theorem (Additive Perturbation)

Let $A \in \mathbb{C}^{m \times n}$ be a matrix with full column rank ($= n$). Then

$$\min_{\Delta \in \mathbb{C}^{m \times n}} \{\|\Delta\|_2 : A + \Delta \text{ has rank } < n\} = \sigma_{\min}(A).$$

Proof:

- If $A + \Delta$ has rank $< n$, then there exists x, with $\|x\|_2 = 1$, such that $(A + \Delta)x = 0$, i.e., $\Delta x = -Ax$.
- In terms of norms, $\|\Delta\|_2 \geq \|\Delta x\|_2 = \|Ax\|_2 \geq \sigma_{\min}(A)$
- To prove that the bound is tight, let us construct a Δ that achieves it.
  - Choose $\Delta = -\sigma_{\min} u_{\min} v'_{\min}$. Clearly, $\|\Delta\| = \sigma_{\min}$.
  - $(A + \Delta)v_{\min} = \left(\sum_{i=1}^{n} \sigma_i u_i v'_i\right) v_{\min} - \sigma_{\min} u_{\min} v'_{\min} v_{\min} = \sigma_{\min} u_{\min} - \sigma_{\min} u_{\min} = 0$.

Multiplicative Perturbation

Theorem (Small Gain)

Let $A \in \mathbb{C}^{m \times n}$:

$$\min_{\Delta \in \mathbb{C}^{n \times n}} \{\|\Delta\|_2 : (I - A\Delta) \text{ is singular } \} = \frac{1}{\sigma_{\max}(A)},$$

i.e., $(I - A\Delta)$ is non-singular if $\|A\|_2\|\Delta\|_2 < 1$.

Proof:

- If $I - A\Delta$ is singular, then there exists $x \neq 0$, such that $(I - A\Delta)x = 0$.
- Hence, $\|x\|_2 = \|A\Delta x\|_2 \leq \|A\|_2\|\Delta x\|_2 = \sigma_{\max}(A)\|\Delta x\|_2$,
- that is, $\Delta_2 \geq \frac{\|\Delta x\|_2}{\|x\|_2} \geq \frac{1}{\sigma_{\max}(A)}$.
- To show that the bound is tight, choose $\Delta = \frac{1}{\sigma_{\max}(A)} v_{\max} u'_{\max}$. Clearly $\|\Delta\|_2 = 1/\sigma_{\max}(A)$, and pick $x = u_{\max}$.
- Then, $(I - A\Delta)x = u_{\max} - \frac{1}{\sigma_{\max}(A)} A v_{\max} = u_{\max} - u_{\max} = 0$.

Perturbations measured in the Frobenius norm

- A useful inequality: $\|A\|_{\mathrm{F}} \geq \|A\|_2$, for any $A \in \mathbb{C}^{m\times n}$.
  $\|A\|_{\mathrm{F}}^2 = \mathrm{Trace}(A'A) = \sum_{i=1}^{n} \sigma_i^2 \geq \sigma_{\max}^2 = \|A\|_2^2$.
- Note: a rank-one matrix $A_1 = uv' \neq 0$ only has only one non-zero singular value. Hence, its Frobenius norm is equal to its induced 2-norm.
- Since the matrices Δ used in the proofs of the perturbation bounds were both rank-one, the results extends to the Frobenius norm case:

Theorem (Additive Perturbation)

$$\min_{\Delta \in \mathbb{C}^{m\times n}} \{\|\Delta\|_{\mathrm{F}} : A + \Delta \textit{ has rank } < n\} = \sigma_{\min}(A).$$

Theorem (Small Gain)

$$\min_{\Delta \in \mathbb{C}^{n\times n}} \{\|\Delta\|_{\mathrm{F}} : (I - A\Delta) \textit{ is singular } \} = \frac{1}{\sigma_{\max}(A)},$$

Total Least Squares

- In the least squares estimation problem, we considered an inconsistent system of equations $y = Ax$ (where A has more rows than columns).
- In order to compute a solution, we introduced a notion of "measurement error" $e = y - Ax$, and looked for a solution that is compatible with the smallest measurement error.
- A more general model (total least squares) also considers a notion of "modeling error," i.e., looks for a solution x of

$$y = (A + \Delta)x + e,$$

that minimizes $\|\Delta\|_F + \|e\|_2 = \|[\Delta, e]\|_F$.

Total Least Squares Solution

- Rewrite the problem in block matrix form:

$$\min_{\|\Delta\|_F + \|e\|_2} \left(\begin{bmatrix} A & -y \end{bmatrix} + \begin{bmatrix} \Delta & e \end{bmatrix} \right) \begin{bmatrix} x \\ 1 \end{bmatrix} = 0,$$

i.e.,

$$\min_{\|\hat{\Delta}\|_F} \left(\hat{A} + \hat{\Delta} \right) \hat{x} = 0,$$

- For this problem to have a valid solution, $\hat{A} + \hat{\Delta}$ must be singular ($\hat{x} \neq 0$).
- This is an additive perturbation problem, in the Frobenius norm... we know the smallest perturbation is $\hat{\Delta} = -\sigma_{\min}(\hat{A}) u_{\min} v'_{\min}$.
- The total least squares solution is obtained by $\hat{x} = v_{\min}$, rescaled so that the last entry is equal to 1, i.e., $\begin{bmatrix} x' & 1 \end{bmatrix} = \alpha v'_{\min}$.

Conditioning of Matrix Inversion

- Consider the matrix $A = \begin{bmatrix} 100 & 100 \\ 100.2 & 100 \end{bmatrix}$. Its inverse is $A^{-1} = \begin{bmatrix} -5 & 5 \\ 5.01 & -5 \end{bmatrix}$.
- Consider the perturbed matrix $A + \delta A = \begin{bmatrix} 100 & 100 \\ 100.1 & 100 \end{bmatrix}$. Its inverse is $(A + \delta A)^{-1} = \begin{bmatrix} -10 & 10 \\ 10.01 & -10 \end{bmatrix}$.
- A 0.1% change in one of the entries of A results in a 100% change in the entries of A^{-1}! Similarly for the solution of linear systems of the form $Ax = y$.
- Under what conditions does this happen? i.e., under what conditions is the inverse of a matrix extremely sensitive to small perturbations in the elements of the matrix?

Condition number

- Differentiate $A^{-1}A = I$. We get $d(A^{-1})\,A + A^{-1}\,dA = 0$.
- Rearranging, and taking the norm:

$$\|d(A^{-1})\| = \| - A^{-1}\; dA\; A^{-1}\| \leq \|A^{-1}\|^2\; \|dA\|$$

- That is,

$$\frac{\|d(A^{-1})\|}{\|A^{-1}\|} \leq \|A^{-1}\|\; \|A\|\; \frac{\|dA\|}{\|A\|}$$

- The quantity $K(A) = \|A^{-1}\|\|A\|$, called the condition number of the matrix A gives a bound on the relative change on A^{-1} given by a perturbation on A.
- If we are considering the induced 2-norm,

$$K(A) = \|A^{-1}\|\|A\| = \sigma_{\max}(A)/\sigma_{\min}(A).$$

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