6.241 Dynamic Systems and Control

Lecture 4: Singular Values

Readings: DDV, Chapter 4

Emilio Frazzoli

Aeronautics and Astronautics
Massachusetts Institute of Technology

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Outline

1. Singular Values
2. Norm computations through singular values

Unitary Matrices

- A square matrix $U \in \mathbb{C}^{n \times n}$ is unitary if $U'U = UU' = I$.
- A square matrix $U \in \mathbb{R}^{n \times n}$ is orthogonal if $U^T U = UU^T = I$.
- Properties:
  - If U is a unitary matrix, then $\|Ux\|_2 = \|x\|_2$, for all $x \in \mathbb{C}^n$.
  - If $S = S'$ is a Hermitian matrix, then there exists a unitary matrix U such that $U'SU$ is a diagonal matrix. [1]
  - For any matrix $A \in \mathbb{R}^{m \times n}$, both $A'A \in \mathbb{R}^{n \times n}$, $AA' \in \mathbb{R}^{m \times m}$ are Hermitian $\Rightarrow$ can be diagonalized by unitary matrices.
  - For any matrix A, the eigenvalues of $A'A$ and AA' are always real [2] and non-negative [3] (in other words, $A'A$ and AA' are positive definite).

[1] $S = S' \Leftrightarrow \langle Sx, y\rangle = \langle x, Sy\rangle$. Let v_1 be an eigenvector of S, and let $M_1 = \mathcal{R}(v_1)^\perp$. If $u \in M_1$, then so is Su: $\langle Su, v_1\rangle = \langle u, Sv_1\rangle = \langle u, \lambda_1 v_1\rangle = 0$. All other eigenvectors must be in M_1. Finite induction gets the result.

[2] Assuming $\langle v_1, v_1\rangle = 1$, $\lambda_1 = \langle Sv_1, v_1\rangle = \langle v_1, Sv_1\rangle = \langle Sv_1, v_1\rangle' = \lambda_1'$

[3] $0 < \langle Av_1, Av_1\rangle = v_1' A' A v_1 = \lambda_1 v_1' v_1$.

Singular Value Decomposition

Theorem (SVD)

Any matrix $A \in \mathbb{C}^{m \times n}$ can be decomposed as $A = U\Sigma V$, where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary matrices. The matrix $\Sigma \in \mathbb{R}^{m \times n}$ is "diagonal," with non-negative elements on the main diagonal. The non-zero elements of Σ are called the singular values of A, and satisfy $\sigma_i = \sqrt{i\text{-th eigenvalue of } A'A}$.

Proof (assuming $\mathrm{rank}(A) = m$):

- Since AA' is Hermitian, there exist a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) > 0$ such that $U\Lambda U' = AA'$.
- Write $\Lambda = \Sigma_1^2 = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2)$
- Define $V_1' := \Sigma_1^{-1} U' A \in \mathbb{R}^{m \times n}$. Clearly, $V_1' V_1 = \Sigma_1^{-1} U' A A' U \Sigma_1^{-1} = I^{m \times m}$.
- Construct $V = [V_1,\ V_2] \in \mathbb{C}^{n \times n}$ by choosing the columns in V_2 so that V is unitary, and $\Sigma = [\Sigma_1,\ 0] \in \mathbb{R}^{n \times n}$, by padding with zeroes.
- Hence, $\Sigma V' = \Sigma_1 V_1' + 0 V_2' = U' A$, i.e., $A = U \Sigma V'$.

Singular Vectors

- If U and V are written as sequences of column vectors, i.e., $U = \left[u_1, u_2, \ldots, u_m\right]$ and $V = \left[v_1, v_2, \ldots, v_n\right]$, then

$$A = U\Sigma V' = \sum_{i=1}^{r} \sigma_i u_i v_i'$$

- The columns of U are called the left singular vectors, and the columns of V are called the right singular vectors.
- Note:
  - Ax can be written as the weighted sum of the left singular vectors, where the weights are given by the projections of x onto the right singular vectors:

$$Ax = \sum_{i=1}^{r} \sigma_i u_i (v_i' x),$$

  - The range of A is given by the span of the first r vectors in U
  - The rank of A is given by r;
  - The nullspace of A is given the span of the last $(n - r)$ vectors in V.

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Induced 2-norm computation

Theorem (Induced 2-norm)

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_{\max}(A).$$

Proof:

$$\sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|U\Sigma V'x\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|\Sigma V'x\|_2}{\|x\|_2} =$$

$$\sup_{y \neq 0} \frac{\|\Sigma y\|_2}{\|Vy\|_2} = \sup_{y \neq 0} \frac{\|\Sigma y\|_2}{\|y\|_2} = \sup_{y \neq 0} \frac{\left(\sum_{i=1}^{n} \sigma_i^2 |y_i|^2\right)^{1/2}}{\left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}} \leq \sigma_{\max}(A).$$

Assuming $\sigma_{\max} = \sigma_1$, the supremum is attained for $y = (1, 0, \ldots, 0)$. This corresponds to $x = v_1$, and $Av_1 = \sigma u_1$

Minimal amplification

Theorem

Given $A \in \mathbb{C}^{m \times n}$, *with* $\mathrm{rank}(A) = n$,

$$\inf_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_n(A).$$

Proof:

$$\inf_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \inf_{x \neq 0} \frac{\|U\Sigma V'x\|_2}{\|x\|_2} = \inf_{x \neq 0} \frac{\|\Sigma V'x\|_2}{\|x\|_2} =$$

$$\inf_{y \neq 0} \frac{\|\Sigma y\|_2}{\|Vy\|_2} = \inf_{y \neq 0} \frac{\|\Sigma y\|_2}{\|y\|_2} = \inf_{y \neq 0} \frac{\left(\sum_{i=1}^{n} \sigma_i^2 |y_i|^2\right)^{1/2}}{\left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}} \geq \sigma_{\min}(A).$$

Assuming $\sigma_{\min} = \sigma_n$, the supremum is attained for $y = (0, \ldots, 0, 1)$. This corresponds to $x = v_n$, and $Av_n = \sigma u_n$

Frobenius norm computation

Theorem

$$\|A\|_{\mathrm{F}} = \left(\sum_{i=1}^{r} \sigma_i(A)^2 \right)^{1/2}$$

Proof:

$$\|A\|_{\mathrm{F}} = \left(\sum_{j=1}^{n} \sum_{i=1}^{m} |a_{ij}|^2 \right)^{1/2} = \left(\mathrm{Trace}(A'A)\right)^{1/2} = \left(\mathrm{Trace}(V\Sigma' U' U \Sigma V')\right)^{1/2} =$$

$$\left(\mathrm{Trace}(V'V\Sigma^2)\right)^{1/2} = \left(\mathrm{Trace}(\Sigma^2)\right)^{1/2} = \left(\sum_{i=1}^{r} \sigma_i^2 \right)^{1/2}$$

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