6.241 Dynamic Systems and Control

Lecture 3: Least Square Solutions of Linear Problems

Readings: DDV, Chapter 3

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Outline

Least Squares Control

- Consider a system of m equations in n unknowns, with $m < n$, of the form[1]

$$y = A'x.$$

- In these conditions, there are in general many possible values for x that make $A'x = y$. It may be of interest to find such a solution x of minimum (weighted) Euclidean norm, i.e., of minimum $\|x\|$.

Example

This situation occurs often in the context of control. For example, consider the simple discrete-time system $x[i+1] = ax[i] + bu[i]$, with initial condition $x[0] = 0$. It is desired to bring this system to $x[N] = y$, while minimizing the cost $\sum_{i=1}^{N} |u[i]|^2$. This problem can be written as

$$\min_u \|u\|^2, \qquad \text{s.t. } y = [b, ab, a^2, ...][u[N-1], u[N-2], \ldots]'$$

[1]We use the notation $y = A'x$ so that A is still a "tall" matrix columns are possibly infinite-dimensional.

Constructing all possible solutions

- It turns out that a $\check{x} = A(A'A)^{-1}y$ is a solution. (Check by substitution).
- However let the null space of A', $\mathcal{N}(A')$, be the subspace containing all solutions of the homogeneous equation $A'x = 0$
- Then, if $\check{x}$ is a solution of $y = A'x$, then so is $\check{x} + x_h$, where $x_h \in \mathcal{A}$.
- In addition, should x_1 and x_2 be two solutions of $y = A'x$, then their difference must be in $\mathcal{N}(A)$.
- The trick is again that of finding among all such solutions the one with minimum norm—which is in fact $\check{x}$.

A general formulation

- Consider the formulation of the problem in terms of Gram product: $y = \prec A, x \succ$.
- Let x be a solution of $y = \prec A, x \succ$. Decompose it into its projection onto $\mathcal{R}(A)$ and on its orthogonal component, i.e., $x = x_A + x_{A^\perp}$.
- Hence $x_A = A\alpha$, $x_{A^\perp} = x - x_A$. Since it must be

$$\langle a_i, x - A\alpha \rangle = 0, \quad i = 1, \ldots, n,$$

i.e.,

$$\prec A, x - A\alpha \succ = 0.$$

With some algebra, we get

$$\alpha = \prec A, A \succ^{-1} \prec A, x \succ,$$

and finally

$$x_A = A \prec A, A \succ^{-1} \prec A, x \succ = A \prec A, A \succ^{-1} y.$$

- Since $\|x\|^2 = \|x_A\|^2 + \|x_{A^\perp}\|^2$, it is clear that one would choose $x_{A^\perp} = 0$, and hence $\check{x} = x_A$.

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