6.241 Dynamic Systems and Control

Lecture 1: Introduction, linear algebra review

Readings: DDV, Chapter 1

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February 2, 2011

Outline

Course Objectives

- The course addresses **dynamic systems**, i.e., systems that evolve with time. Typically these systems have inputs and outputs: it is of interest to understand how the input affects the output (or, vice-versa, what inputs should be given to generate a desired output).
- In particular, we will concentrate on systems that can be modeled by **Ordinary Differential Equations (ODEs)**, and that satisfy certain **linearity** and **time-invariance conditions**. In general, we will consider systems with multiple inputs and multiple outputs (MIMO).
- We will analyze the response of these systems to inputs and initial conditions: for example, **stability** and **performance** issues will be addressed. It is of particular interest to analyze systems obtained as interconnections (e.g., feedback) of two or more other systems.
- We will learn how to **design** (control) systems that ensure desirable properties (e.g., stability, performance) of the interconnection with a given dynamic system.

Course Outline

The course will be structured in several major sections:

- A review of linear algebra, and of least squares problems.
- Representation, structure, and behavior of multi-input, multi-output (MIMO) linear time-invariant (LTI) systems.
- Robust Stability and Performance. Approaches to optimal and robust control design.

Hopefully, the material learned in this course will form a valuable foundation for further work in systems, control, estimation, identification, signal processing, and communications.

Assignments

Homework Generally handed out every Wednesday, and due in class a week later (except as noted on schedule), at which time solutions will be handed out.

Tests There will be two exams:

- Midterm Exam, March 16, TBC (take home?)
- Final Exam (during final exam week)

Grading The course grade will depend on: (a) your involvement in the subject (30%), as evidenced mainly by your homework, but also by your interaction with the TAs and instructor; (b) your performance on the the midterm exam (30%), and the final exam (40%).

Notes and Texts

There is *no required text*. Lecture notes are *required* and available in the Readings section of the OCW site.

Other texts that you may wish to examine at some point are

- D.G. Luenberger, *Introduction to Dynamic Systems*, Wiley, 1979.
- T. Kailath, *Linear Systems*, Prentice-Hall, 1980.
- J.C. Doyle, B.A. Francis, and A.R. Tannenbaum, *Feedback Control Theory*, Macmillan, 1992. (Available on the OCW site.)
- R.J. Vaccaro, *Digital Control: A State-Space Approach*, McGraw-Hill, 1995.

Tentative schedule

#	Date	Topic	Chapter
1	Feb 2, 2011	Introduction to dynamic systems and control. Matrix algebra.	Ch 1
2	Feb 7, 2011	Least Squares error solutions of overdetermined/underdetermined systems	Ch 2, 3
3	Feb 9, 2011	Matrix Norms, SVD, Matrix perturbations	Ch 4
4	Feb 14, 2011	Matrix Perturbations	Ch 5
5	Feb 16, 2011	State-space models, Linearity and time invariance	Ch 6,7,8
6	Feb 22, 2011	Solutions of State-space models	10, 11
7	Feb 23, 2011	Similarity transformations, modes of LTI systems, Laplace transform, Transfer functions	12
8	Feb 28, 2011	Stability, Lyapunov methods	13, 14
9	Mar 2, 2011	External I/O stability, Storage functions	15
10	Mar 7, 2011	Interconnected Systems, Feedbck, I/O Stability	15, 17
11	Mar 9, 2011	System Norms	16
12	Mar 14, 2011	Performance Measures in Feedback Control	18
13	Mar 16, 2011	Small Gain Theorem, stability robustness	19

Tentative schedule

#	Date	Topic	Chapter
14	Mar 28, 2011	Stability Robustness (MIMO)	20, 21
15	Mar 30, 2011	Reachability	22
16	Apr 4, 2011	Reachability - standard and canonical forms, modal tests	23
17	Apr 6, 2011	Observability	24
18	Apr 11, 2011	Minimality, Realization, Kalman Decomposition, Model reduction	25
19	Apr 13, 2011	State feedback, observers, output feedback, MIMO poles and zeros	26-29
20	Apr 20, 2011	Minimality of interconnections, pole/zero cancellations	30
21	Apr 25, 2011	Parameterization of all stabilizing controllers	
22	Apr 27, 2011	Optimal control synthesis: problem setup	
23	May 2, 2011	H_2 optimization	
24	May 4, 2011	H_∞ optimization	
25	May 9, 2011	TBD	
26	May 11, 2011	TBD	

Outline

1. Syllabus review

2. Linear Algebra Review

Vector Spaces

A vector space is defined as a set V over a (scalar) field F, together with two binary operations, i.e., vector addition $(+)$ and scalar multiplication $(\cdot)$, satisfying the following axioms:

- Commutativity of $+$: $u + v = v + u, \forall u, v, \in V$;
- Associativity of $+$: $u + (v + w) = (u + v) + w, \forall u, v, w \in V$;
- Identity element for $+$: $\exists 0 \in V : v + 0 = 0 + v = v, \forall v \in V$;
- Inverse element for $+$: $\forall v \in V \exists (-v) \in V : v + (-v) = (-v) + v = 0$;
- Associativity of $\cdot$: $a(bv) = (ab)v, \forall a, b \in F, v \in V$;
- Distributivity of $\cdot$ w.r.t. vector $+$: $a(v + w) = av + aw, \forall a \in F, v, w \in V$;
- Distributivity of $\cdot$ w.r.t. scalar $+$: $(a + b)v = av + bv, \forall a, b \in F, v \in V$;
- Normalization: $1v = v, \forall v \in V$.

Vector space examples (or not?)

- $\mathbb{R}^n$, $\mathbb{C}^n$;
- Real continuous functions $f : \mathbb{R} \to \mathbb{R}$
- The set of $m \times n$ matrices;
- The set of solutions $y(t)$ of the LTI ODE $dy(t)/dt + 3y(t) = 0$;
- The set of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ satisfying $x_1^2 + x_2^2 + x_3^2 = 1$.
- The set of solutions $y(t)$ of the LTI ODE $dy(t)/dt + 3y(t) = 0$.

Subspaces

- A **subspace** of a vector space is a subset of vectors that itself forms a vector space.
- A necessary and sufficient condition for a subset of vectors to form a subspace is that this subset be closed with respect to vector addition and scalar multiplication.

Subspace examples (or not?)

- The range on any real $n \times m$ matrix, and the nullspace of any $m \times n$ matrix.
- The set of all linear combinations of a given set of vectors.
- The intersection of two subspaces.
- The union of two subspaces.
- The Minkowski (or direct) sum of two subspaces.

Linear (in)dependence, bases

- n vectors $v_1, v_2, \ldots, v_n \in V$ are (linearly) independent if

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0 \quad \Leftrightarrow \quad c_1, c_2, \ldots, c_n = 0.$$

- A space is n-dimensional if every set of more than n vectors is dependent, but there is some set of n vectors that are independent.
- Any set of n independent vectors is also called a basis for the space.
- if a space contains a set of n independent vectors for any $n \in \mathbb{N}$, then the space is infinite-dimensional.

Norms

Norms measure the ‘length” of a vector. A norm maps all vectors in a vector space to a non-negative scalar, with the following properties:

- Positivity: $\|x\| > 0$ for $x \neq 0$.
- Homogeneity: $\|ax\| = |a| \, \|x\|, \quad \forall a \in \mathbb{R}, x \in V.$
- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$.

Norm examples (or not?)

- Usual Euclidean norm in $\mathbb{R}^n$, $\|x\| = \sqrt{x'x}$;
  (where x' is the conjugate transpose of x, i.e., as in Matlab).
- A matrix Q is Hermitian if $Q' = Q$, and positive definite if $x'Qx > 0$ for $x \neq 0$. Then $\|x\| = \sqrt{x'Qx}$ is a norm.
- For $x \in \mathbb{R}^n$, $\|x\|_1 = \sum_1^n |x_i|$, and $\|x\|_\infty = \max_i |x_i|$.
- For a continuous function $f : [0,1] \to \mathbb{R}$:
  $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$, and $\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}$.

Inner product

- An inner product on a vector space V (with scalar field F) is a binary operation $\langle \cdot, \cdot \rangle : V \times V \to F$, with the following properties:
  - Symmetry: $\langle x, y \rangle = \langle y, x \rangle'$, $\forall x, y \in V$;
  - Linearity: $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$;
  - Positivity: $\langle x, x \rangle > 0$ for $x \neq 0$.
- The inner product gives a geometric structure to the space; e.g., it allows to reason about angles, and in particular, it defines orthogonality. Two vectors x and y are orthogonal if $\langle x, y \rangle = 0$.
- Let $S \subseteq V$ be a subspace of V. The set of vectors orthogonal to all vectors in S is called $S^{\perp}$, the orthogonal complement of S, and is itself a subspace.

Inner product and norms

- An inner product induces a norm $\|x\| = \sqrt{\langle x, x \rangle}$.
- For example, define $\langle x, y \rangle = x'Qy$ with Q Hermitian positive definite.
- For f, g continuous functions on $[0, 1]$, let $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$
- Cauchy-Schwartz inequality: $|\langle x, y \rangle| \leq \|x\| \, \|y\|$, $\forall x, y \in V$,
  with equality only if $y = \alpha x$ for some $\alpha \in F$.
  (assuming that the norm is that induced by the inner product)

  Proof

$$0 \leq \langle x + \alpha y, x + \alpha y \rangle = x'x + \alpha' y' x + \alpha x' y + |\alpha|^2 y' y$$

$$\text{Choose } \alpha = -x'y / \langle y, y \rangle :$$

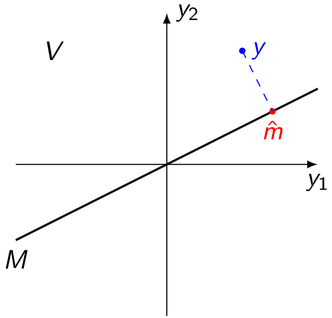
$$0 \leq \langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2.$$

The Projection Theorem

- Let M be a subspace of an inner product space V. Given some $y \in V$, consider the following minimization problem:

$$\min_{m \in M} \|y - m\|,$$

where the norm is that induced by the inner product in V.



Projection theorem

The optimal solution $\hat{m}$ is such that $(y - \hat{m}) \perp M$

Proof of the projection theorem

- By contradiction: assume that $y - \hat{m}$ is not orthogonal to M, i.e., there is some m_0, $\|m_0\| = 1$, such that $\langle y - \hat{m}, m_0 \rangle = \delta \neq 0$.

- Then argue that $(\hat{m} + \delta' m_0) \in M$ achieves a better solution than $\hat{m}$. In fact:

$$\begin{aligned} \|y - \hat{m} - \delta' m_0\|^2 &= \|y - \hat{m}\|^2 - \delta' \langle y - \hat{m}, m_0 \rangle - \delta \langle m_0, y - \hat{m} \rangle + |\delta|^2 \|m_0\|^2 \\ &= \|y - \hat{m}\|^2 - |\delta|^2 - |\delta|^2 + |\delta|^2 \|m_0\|^2 = \|y - \hat{m}\|^2 - |\delta|^2. \end{aligned}$$

Linear Systems of equations

- Consider the following system of (real or complex) linear equations:

$$Ax = y, \qquad A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

- Given A and y, is there a solution x?

$$\exists \text{ a solution } x \quad \Leftrightarrow \quad y \in \mathcal{A} \quad \Leftrightarrow \quad \mathcal{R}([A|y]) = \mathcal{R}(A).$$

- There are three cases:
  1. $n = m$: if $\det(A) \neq 0) \Rightarrow x = A^{-1}y$ is the unique solution.
  2. $m > n$: more equations than unknowns, the system is overconstrained. Happens in, e.g., estimation problems, where one tries to estimate a small number of parameters from a lot of experimental measurements. In such cases the problem is typically inconsistent, i.e., $y \notin \mathcal{R}(A)$. So one is interested in finding the solution that minimizes some error criterion.
  3. $m < n$: more unknown than equations, the system is overconstrained. Happens in, e.g., control problems, where there may be more than one way to complete a desired task. If there is a solution x_p (i.e., $Ax_p = y$), then typically there are many other solutions of the form $x = x_p + x_h$, where $x_h \in \mathcal{N}(A)$ (i.e., $Ax_h = 0$). In this case it is desired to find the solution than minimizes some cost criterion.