MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Spring 2011

Homework 8 Solutions

Exercise 17.4 1) First, in order for the closed loop system to be stable, the transfer function from $\begin{pmatrix} w_1 & w_2 \end{pmatrix}^T$ to $\begin{pmatrix} y & u \end{pmatrix}^T$ has to be stable. The transfer function from w_1 to y is given by $(I - PK)^{-1}P$ and is called system response function. The transfer function from w_1 to u is given by $(I - KP)^{-1}$ and is called input sensitivity function. The transfer function from w_2 to y is $(I - PK)^{-1}PK$ and is called the complementary sensitivity function. The transfer function from w_2 to u is given by $(I - KP)^{-1}K$. Therefore, we have the following :

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} (I+PK)^{-1}P & (I+PK)^{-1}PK \\ (I+KP)^{-1} & (I+KP)^{-1}K \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

So, if K is given as

$$K = Q(I - PQ)^{-1} = (I - QP)^{-1}Q,$$

then

$$\begin{aligned}
(I+PK)^{-1}P &= (I + PQ(I-PQ)^{-1})^{-1}P \\
&= (((I-PQ) + PQ)(I-PQ)^{-1})^{-1}P \\
&= (I-PQ)P \\
(I+PK)^{-1}PK &= (I-PQ)PQ(I-PQ)^{-1} \\
&= P(I-QP)(I-QP)^{-1}Q \\
&= PQ \\
(I+KP)^{-1} &= (I + (I-QP)^{-1}QP)^{-1} \\
&= ((I-QP+QP)(I-QP)^{-1})^{-1} \\
&= I - QP \\
(I+KP)^{-1}K &= (I-QP)(I-QP)^{-1}Q \\
&= Q.
\end{aligned}$$

Thus, the closed loop transfer function can be now written as follows:

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} (I-PQ)P & PQ \\ (I-QP) & Q \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

In order for the closed loop system to be stable, then all the transfer functions in the large matrix above must be stable as well.

$$\begin{aligned}
\|(I-PQ)P\| &\le \|I-PQ\|\|P\| \le (\|I\| + \|PQ\|)\|P\| \\
&\le (\|I\| + \|P\|\|Q\|)\|P\| \le \|P\| + \|P\|^2\|Q\| \\
\|PQ\| &\le \|P\|\|Q\| \\
\|I-QP\| &\le \|I\| + \|QP\| \le \|I\| + \|Q\|\|P\|.
\end{aligned}$$

Since P and Q are stable from the assumptions, we know that all the transfer functions are stable. Therefore the closed loop system is stable if $K = Q(I - PQ)^{-1} = (I - QP)^{-1}Q$.

2) From 1), we can express Q in terms of P and K in the following manner.

$$\begin{aligned} K &= Q(I - PQ)^{-1} \\ K(I - PQ) &= Q \\ K - KPQ &= Q \\ K &= (I + KP)Q \\ \rightarrow Q &= (I + KP)^{-1}K = K(I + PK)^{-1}, \end{aligned}$$

by push through rule.

For *some* stable Q, the closed loop is stable for a stable P. by the stabilizing controller $K = Q(I - PQ)^{-1}$. Yet, *not all* stable Q can be used for this formulation because of the well-posedness of the closed loop. In the state space descriptions of P and Q, in order for the interconnected system, in this case $K(s)$ to be well-posed, we have to have the condition (17.4) in the lecture note, i.e., $(I - D_P Q(\infty))$ is invertible.

3) Suppose P is SISO, w_1 is a step, and $w_2 = 0$. Then, we have the following closed loop transfer function:

$$\begin{pmatrix} Y(s) \\ U(s) \end{pmatrix} = \begin{pmatrix} (I - PQ)P \\ I - QP \end{pmatrix} \frac{1}{s},$$

since the Laplace transform of the unit step is $\frac{1}{s}$ we have

$$U(s) = (1 - Q(s)P(s))\frac{1}{s}.$$

Then using the final value theorem, in order to have the steady state value of $u(\infty)$ to be zero, we need:

$$\begin{aligned} u(\infty) &= \lim_{s \to 0} s(1 - Q(s)P(s))\frac{1}{s} = 0 \\ \rightarrow 1 - Q(0)P(0) &= 0 \\ \rightarrow Q(0) &= 1/P(0). \end{aligned}$$

Therefore, $Q(0)$ must be nonzero and is equal to $1/P(0)$. Note that this condition implies that P cannot have a zero at $s = 0$ because then Q would have a pole at $s = 0$, which contradicts that Q is stable.

Exercise 17.5 a) Let $l(s)$ be the signal at the output of $Q(s)$, then we have

$$\begin{aligned} l &= Q(r - (P - P_0)l) \\ \rightarrow (I + Q(P - P_0))l &= Qr \\ \therefore l &= (I + Q(P - P_0))^{-1}Qr. \end{aligned}$$

Since we can write $y = Pl$, and with $P(s) = \frac{2}{s-1}$, $P_0(s) = \frac{1}{s-1}$, and $Q = 2$, the transfer function from r to y can be calculated as follows:

$$
\begin{aligned}
Y(s) &= P(s)L(s) \\
&= P(I + Q(P - P_0))^{-1} Q R(s) \\
&= \frac{2}{s-1}\left(1 + 2\left(\frac{2}{s-1} - \frac{1}{s-1}\right)\right)^{-1} 2R(s) \\
&= \frac{4}{s-1}\left(\frac{s+1}{s-1}\right)^{-1} R(s) \\
&= \frac{4}{s-1}\frac{s-1}{s+1} R(s) \\
\therefore \frac{Y(s)}{R(s)} &= \frac{4}{s+1}.
\end{aligned}
$$

b) There is an unstable pole/zero cancellation so that the system is not internally stable.
c) Suppose $P(s) = P_0(s) = H(s)$ for some $H(s)$. Then using a part of the equation in a), we have

$$
\begin{aligned}
Y(s) &= H(s)(I + Q(s)(H(s) - H(s)))^{-1} Q(s)R(s) \\
&= H(S)I^{-1}Q(s)R(s) \\
&= H(s)Q(s)R(s) \\
\rightarrow \frac{Y(s)}{R(s)} &= H(s)Q(s).
\end{aligned}
$$

Therefore in order for the system to be internally stable for any $Q(s)$, $H(s)$ has to be stable.

Exercise 19.2 The characteristic polynom for the closed loop system is given by

$$s(s+2)(s+a) + 1 = 0$$

Computing the locus of the closed poles as a function of a can be done numerically. The closed loop system is stable if $a \geq 0.225$. The above bound can also be derived by means of root locus techniques or by evaluating the Routh Hurwitz criterion. Another way of deriving bounds for the value of a is by casting this parametric uncertainty problem as an additive or multiplicative perturbation problem, see also 19.5. One can expect that the derived bounds in such a case would be rather conservative.

Exercise 19.4 We can represent an uncertainty in feedback configuration, as shown below.

Note that the plant is SISO, and we consider blocks Δ and W to be SISO systems as well, so we can commute them. The transfer function seen by the Δ block can be derived as follows:

$$
\begin{aligned}
z &= P_0(-Ww - Kz) \\
&= -(I + P_0K)^{-1} P_0 W w \\
\therefore M &= -(I + P_0K)^{-1} P_0 W.
\end{aligned}
$$

Apply the small gain theorem, and obtain the condition for stability robustness of the closed loop system as follows:

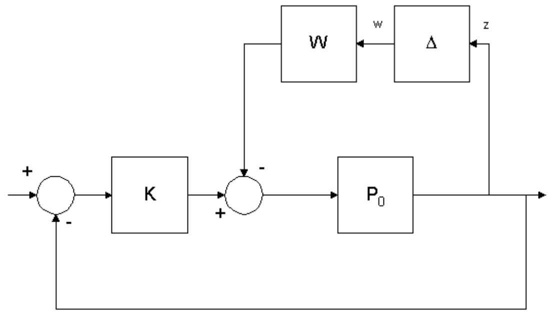

Figure 19.4

$$\sup_{\omega} \left| \frac{W(j\omega)P_0(j\omega)}{1 + P_0(j\omega)K(j\omega)} \right| < 1$$

Exercise 19.5 a) Given

$$P(s) = \frac{1}{s-a}\ ,\ K(s) = 10.$$

In order for the system to remain stable, the zeros of $1 + PK$ be in open left half plane. Thus,

$$1 + PK = 1 + \frac{10}{s-a} = \frac{s - s + 10}{s - a}\ \rightarrow\ a < 10.$$

b) Assume that the nominal plant is $P_0 = \frac{1}{s}$. With $W = -a$,

$$\Omega : \frac{P_0}{1 + W\Delta P_0} = \frac{\frac{1}{s}}{1 - a\Delta\frac{1}{s}} = \frac{1}{s - a\Delta},$$

so when $\Delta = 1$, we have

$$\frac{P_0}{1 + W\Delta P_0} = \frac{1}{s-a} = P,$$

which says that P is clearly in Ω.

c) The transfer function seen by the Δ block was derived is (from the previous problem):

$$M \quad = \quad -(I + P_0K)^{-1}P_0W.$$

Applying the small gain theorem:

$$
\begin{aligned}
\|M\|_\infty &= \sup_\omega |(1+P_0K)^{-1}P_0K| < 1 \\
\rightarrow & \quad \sup_\omega |\frac{P_0W}{1+P_0K}| < 1 \\
\rightarrow & \quad \sup_\omega |\frac{\frac{-a}{j\omega}}{1+\frac{10}{j\omega}}| < 1 \\
& \vdots \\
|a| &< \sqrt{\omega^2+100} \\
\therefore |a| &< 10.
\end{aligned}
$$

Since Δ block can have arbitrary phase we obtained much more conservative constraint on parameter a than the one in a).

d) Now, the nominal plant is replaced by $P_0 = \frac{1}{s+100}$. With the same description of Ω set, first in order to show that $P \in \Omega$ with $P_0 = \frac{1}{s+100}$, we need to find a new W as follows:

$$
\frac{P_0}{1+W\Delta P_0} = \frac{\frac{1}{s+100}}{1+W\Delta\frac{1}{s+100}} = \frac{1}{s+100+W\Delta},
$$

with $\Delta = 1$, the denominator becomes $s + 100 + W$, which we want to equate to $s - a$. Thus we have a new W to be

$$
W = -a - 100.
$$

Then in order to derive the condition on the closed loop system to be stable n the set Ω, we use the small gain theorem again.

$$
\begin{aligned}
\|M\|_\infty &= \sup_\omega |(1+P_0K)^{-1}P_0K| < 1 \\
& \rightarrow \sup_\omega |\frac{P_0W}{1+P_0K}| < 1 \\
& \rightarrow \sup_\omega |\frac{\frac{-a-100}{j\omega+100}}{1+\frac{10}{j\omega+100}}| < 1 \\
|a+100| &< \sqrt{\omega^2+110^2} \\
& \vdots \\
\rightarrow -210 &< a < 10.
\end{aligned}
$$

We can see that by representing uncertainty in a different way we can get a less conservative result.

6.241J / 16.338J Dynamic Systems and Control
Spring 2011