

6.241: Dynamic Systems—Fall 2008

HOMWORK 6 SOLUTIONS

Exercise 13.1 (a) The system described by

$$\begin{aligned}\dot{x} &= z \\ \dot{z} &= -4x^3 + 2\alpha x\end{aligned}$$

has an equilibrium point at $(0, 0)$ for any value of α (In particular, for α in the interval $-1 \leq \alpha \leq 1$). Additionally, when $\alpha > 0$ there are two more equilibrium points: $(0, \pm\sqrt{\frac{\alpha}{2}})$ (In particular, this is the case for α in the interval $0 < \alpha \leq 1$).

(b) Linearizing the system around $(0, 0)$ we get the Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 2\alpha & 0 \end{pmatrix}$$

The characteristic polynomial of the system is $\det(A - \lambda I) = \lambda^2 - 2\alpha$. If $\alpha > 0$ there is an unstable root, hence the linearized model is unstable around $(0, 0)$. If $\alpha < 0$ both roots are on imaginary axis, and the linearized system is marginally stable (neither asymptotically stable nor unstable). To analyze stability of the original non-linear system in this case we would have to look at the higher order terms.

For the two other equilibrium points (which exist for $\alpha > 0$) we get the Jacobian:

$$\begin{pmatrix} 0 & 1 \\ -4\alpha & 0 \end{pmatrix}$$

The characteristic polynomial for the system is $\det(A - \lambda I) = \lambda^2 + 4\alpha$. The complex conjugate roots lie on the $j\omega$ axis and the linearized system is marginally stable.

Exercise 13.2 a) Notice that the input-output differential equation can be written as

$$\ddot{y} = (\dot{u} - a_1\dot{y}) + (u - a_2y - cy^2)$$

and we can use “observability-like” realization employed for a discrete-time system of exercise 7.1 (c). The differential equations for the states are

$$\begin{aligned}\dot{x}_1 &= -a_1x_1 + x_2 + u \\ \dot{x}_2 &= -a_2x_1 - cx_1^2 + u\end{aligned}$$

and the output equation is $y = x_1$. You can check that it is indeed a correct realization by differentiating the first state equation and plugging in an expression for \dot{x}_2 from the second equation.

b) Let us consider the system with zero input and $a_1 = 3$, $a_2 = 2$ and $c = 2$.

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= -2x_1 - 2x_1^2 \end{aligned}$$

The linearized system's matrix is

$$A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$$

with the characteristic equation $\lambda^2 + 3\lambda + 2 = 0$, which has the roots $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore the linearized system is asymptotically stable around the origin, which also means that the original nonlinear system is a.s. around the origin (see lecture notes for the relevant theorem). You can also verify by linearization that the other equilibrium point $(-1, -3)$ is unstable.

c) Let us find a Lyapunov function for the linear system, and then find a region where its derivative is negative definite along the trajectories of the original nonlinear system. Since the linear system is asymptotically stable, for any symmetric positive definite matrix Q there exists a unique positive definite matrix P such that $A'P + PA = -Q$. Let us choose $Q = I$. Solving the system of linear equations imposed by the matrix relation we obtain

$$P = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

which gives us a quadratic Lyapunov function

$$V(x) = x'Px = \frac{1}{2}x_1^2 - x_1x_2 + x_2^2$$

Taking a derivative of $V(x)$ along the trajectory using the chain rule we get:

$$\dot{V}(x) = -x_1^2 - x_2^2 - 2x_1^2(2x_2 - x_1) = -x_1^2(1 + 2(2x_2 - x_1)) - x_2^2$$

The contour lines of the found Lyapunov function are

$$\frac{1}{2}x_1^2 - x_1x_2 + x_2^2 = C^2$$

for various constants C . Let us find such C that if the point $(x_1 \ x_2)$ is within the boundary then $\dot{V}(x) < 0$. Then the trajectory started in this set will stay there, and will asymptotically decay to zero. Note that

$$V(x) = \frac{1}{4}x_1^2 + \frac{1}{4}(x_1 - 2x_2)^2 = C^2$$

therefore

$$|2x_2 - x_1| < 2C$$

Therefore if $C < 1/4$ the derivative is strictly less than zero. Hence we found a region of attraction as an ellipse, given by

$$x_1^2 + (x_1 - 2x_2)^2 < \frac{1}{4}$$

Any ball located completely within this ellipse will also be a region of attraction. Note also that this set is not exhaustive, there are other points in space that converge to zero.

Exercise 14.2 (a) The system is asymptotically stable if all the roots of the characteristic polynomial lie in the left half of the complex plane. Note that characteristic polynomial for matrix A in a control canonical form is given by

$$\det(A - \lambda I) = \lambda^N + a_0\lambda^{N-1} + \dots + a_{N-1}$$

b) Use continuity argument to prove that destabilizing perturbation with the smallest Frobenius norm will place an eigenvalue of $A + \Delta$ on the imaginary axis. Suppose that the minimum perturbation is $\bar{\Delta}$. Assume that there is an eigenvalue in the right half plane. Consider a perturbation of the form $c\bar{\Delta}$, where $0 \leq c \leq 1$. As c changes from 0 to 1 at least one eigenvalue had to cross $j\omega$ axis, and the resulting perturbation has a smaller Frobenius norm than $\bar{\Delta}$. This proves contradiction with the original assumption that $A + \bar{\Delta}$ has an eigenvalue in the right half plane.

c) The characteristic polynomial for the perturbed matrix is

$$\det(A - \lambda I) = \lambda^N + (a_0 + \delta_0)\lambda^{N-1} + \dots + (a_{N-1} + \delta_{N-1})$$

We know that there exists a root $\lambda = j\omega$, where ω is real. If we plug this solution in, and assemble the real and imaginary parts and set them equal to zero, we'll get two linear equations in δ with coefficients dependent on a_k and powers of ω . For example, for a 4th order polynomial:

$$(j\omega)^4 + (a_0 + \delta_0)(j\omega)^3 + (a_1 + \delta_1)(j\omega)^2 + (a_2 + \delta_2)j\omega + a_3 + \delta_3 = 0$$

results in the following two equations:

$$\begin{aligned} \omega^4 - (a_1 + \delta_1)\omega^2 + a_3 + \delta_3 &= 0 \\ -(a_0 + \delta_0)\omega^3 + (a_2 + \delta_2)\omega &= 0 \end{aligned}$$

This equation can be written in matrix form as follows:

$$\begin{pmatrix} 0 & -\omega^2 & 0 & 1 \\ -\omega^3 & 0 & \omega & 0 \end{pmatrix} \begin{pmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} -\omega^4 + a_1\omega^2 - a_3 \\ a_0\omega^3 - a_2\omega \end{pmatrix}$$

or

$$A(\omega)\delta = B(\omega)$$

Therefore the problem can be formulated as finding a minimal norm solution to an underdetermined system of equations:

$$\min_{A(\omega)\delta=B(\omega)} \|\delta\|$$

By inspection we can see that matrix A has full row rank for any value of ω unequal to zero. If $\omega = 0$ the solution is $\delta_3 = -a_3$, and the rest of the δ_k equal to zeros. For all other values of ω the solution can be expressed as a function of ω :

$$\bar{\delta}(\omega) = A'(\omega) [A(\omega)A'(\omega)]^{-1} B(\omega)$$

Note that the matrix AA' is diagonal, and can be easily inverted. By minimizing the norm of this expression over ω we can find $\bar{\omega}$ that corresponds to the minimizing perturbation. Then plug this $\bar{\omega}$ in the previous equation to compute minimizing perturbation $\bar{\delta}(\bar{\omega})$ explicitly. Compare the norms of the solutions corresponding to $\omega = \bar{\omega}$ and $\omega = 0$ (i.e. compare $\|\bar{\delta}(\bar{\omega})\|$ and a_3), and choose the

minimum as the solution. This way we converted the problem to minimization of a function of a single variable, which can be easily solved.

d) In case $N = 2$ the characteristic polynomial of the perturbed matrix is

$$\lambda^2 + (a_0 + \delta_0)\lambda + (a_1 + \delta_1) = 0$$

where $\lambda = j\omega$. For $\omega = 0$ the minimizing solution is $\delta_1 = -a_1$, $\delta_0 = 0$. If $\omega \neq 0$, plug in $\lambda = j\omega$, and the resulting system of equations is

$$\begin{aligned}\delta_1 &= \omega^2 - a_1 \\ \delta_0 &= -a_0\end{aligned}$$

This is a proper system (number of equations is equal to the number of unknowns), and its solution is given directly by the equations. To minimize the norm of the solution we set $\omega = \sqrt{a_1}$. Note that stability of original matrix A requires that $a_1 > 0$, $a_0 > 0$ (in fact positivity of all coefficients is always a necessary condition, but not sufficient for $N > 2$ - use Routh criterion for a test in that case!). Next, we have to compare $|a_1|$ and $|a_0|$, and choose the smallest of them to null with δ_1 or δ_0 . In our problem $a_0 = a_1 = a$, therefore there are 2 solutions: $(0, a)$ and $(a, 0)$ for the set of δ 's.

Exercise 14.5 a) Consider a nonlinear time-invariant system

$$\dot{x} = f(x, u)$$

and its linearization around an equilibrium point (\tilde{x}, \tilde{u}) is

$$\dot{\delta x} = A\delta x + B\delta u.$$

1) Since the feedback $\delta u = K\delta x$ asymptotically stabilize the linearized system, all the eigenvalues of the matrix $A + BK$ are in OLHP.

2) Without loss of generality, we can take $(\tilde{x}, \tilde{u}) = (0, 0)$ where the nonlinear system is linearized around. Then,

$$\dot{x} = f(x, u) = \left. \frac{\partial f}{\partial x} \right|_{(\tilde{x}, \tilde{u})=(0,0)} x + \left. \frac{\partial f}{\partial u} \right|_{(\tilde{x}, \tilde{u})=(0,0)} u + \Phi(x, u),$$

where $\Phi(x, u)$ is in the order of x^2 .

If $u = kx$, the above equation can be written as follows:

$$f(x, u) = Ax + Bu + \Phi(x, kx),$$

where

$$\frac{\|\Phi(x, kx)\|}{\|x\|} \rightarrow 0 \text{ as } \|x\| \rightarrow 0.$$

Thus the original nonlinear system is locally stabilized with the control law $u = kx$.

b) Consider the system S_1 :

$$\ddot{y} + \dot{y}^4 + y^2 u + y^3 = 0$$

where u is the input.

1) Let $x_1 = y$ and $x_2 = \dot{y}$, then state space representation of the system S_1 is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1^3 - x_2^2 u - x_2^4 \end{pmatrix}.$$

Thus the unique equilibrium point x^* is found to be $x^* = (0, 0)$, which is independent on u^* .

2) Choose $u^* = 0$. Then the linearized system is

$$\begin{aligned} \dot{\delta x} &= \begin{pmatrix} 0 & 1 \\ -3x_1^2 - 2x_2 u - 4x_2^3 & 0 \end{pmatrix} \Big|_{(x^*, u^*)=(0,0)} \delta x + \begin{pmatrix} 0 \\ -x_2^2 \end{pmatrix} \Big|_{(x^*, u^*)=(0,0)} \delta u \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \delta x + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \delta u \\ \dot{\delta x} &= A\delta x + B\delta u. \end{aligned}$$

Since the eigenvalues of the matrix A are at 0, and the δu term does not enter to the linearized system, the linearized system can not conclude the local stability of the nonlinear system S_1 .

c) Let $u = c - x_2^3$ where c is a function of x_1 and x_2 . Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^2(c - x_2^3) - x_2^4 = -x_1^3 - cx_2^2. \end{aligned}$$

So, it can be considered that this system has a nonlinear spring.

Now, choose a Lyapunov function candidate in the $V(x) = x_1^4 + x_2^2$. Then,

$$\begin{aligned} \dot{V}(x) &= \begin{pmatrix} 4x_1^3 & 2x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -x_1^3 - cx_2^2 \end{pmatrix} \\ &= 4x_1^3 x_2 - 2x_2 x_1^3 - cx_2^3 \\ &= x_2(2x_1^3 - 2cx_2^2) \\ &= 2x_2(x_1^3 - cx_2^2). \end{aligned}$$

In order to make $\dot{V}(x)$ negative definite, we would like to have set

$$\begin{aligned} x_1^3 - cx_2^2 &= -x_2 - \frac{x_1^2}{x_2} \\ \rightarrow c &= \frac{x_1^2 + x_2^2 + x_1^3 x_2}{x_2^3}. \end{aligned}$$

This choice of c makes $\dot{V}(x)$ be

$$\dot{V}(x) = -2x_1^2 - 2x_2^2 < 0 \quad \forall x \neq 0.$$

Thus the following control input

$$u = -2x_2 \left(x_2 + \frac{x_1^2}{x_2} \right) - x_2^3$$

stabilizes the system.

Exercise 14.7 (a) The linearized system is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2x_2 \\ -x_2 & -(x_1 + 1) \end{bmatrix} \Big|_{\mathbf{0}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hence, the matrix A of the linearized system is:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

A has repeated eigenvalues at -1 . Thus, the nonlinear system is locally asymptotically stable about the origin.

(b) The linearized system is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3x_1^2 & 1 \\ 1 & -1 \end{bmatrix} \Big|_{\mathbf{0}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hence, the matrix A of the linearized system is:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

The characteristic polynomial is: $\lambda(1 + \lambda) - 1 = 0$. The eigenvalues are thus $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ (one of the eigenvalues is in the right half plane), and the nonlinear system is unstable about the origin.

(c) The linearized system is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2x_1 & -1 \end{bmatrix} \Big|_{\mathbf{0}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hence, the matrix A of the linearized system is:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

A has repeated eigenvalues at -1 . Thus, the nonlinear system is locally asymptotically stable about the origin.

(d) The linearized system is given by:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 2 & x_2(k) \\ 1 & 1 \end{bmatrix} \Big|_{\mathbf{0}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Hence, the matrix A of the linearized system is:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$. Since one of the eigenvalues is outside the unit disk, the nonlinear system is unstable about the origin.

(e) The linearized system is given by:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -x_2(k)e^{x_1(k)x_2(k)} & -x_1(k)e^{x_1(k)x_2(k)} \\ 1 & 2 \end{bmatrix} \Big|_{\mathbf{0}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Hence, the matrix A of the linearized system is:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

A has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$. Since one of the eigenvalues is outside the unit disk, the nonlinear system is unstable about the origin.

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