MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Spring 2011

Homework 1 Solutions

Exercise 1.1 a) Given square matrices A_1 and A_4, we know that A is square as well:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & A_4 \end{pmatrix} \cdot \begin{pmatrix} A_1 & A_2 \\ 0 & I \end{pmatrix}$$

Note that

$$det \begin{pmatrix} I & 0 \\ 0 & A_4 \end{pmatrix} = det(I)det(A_4) = det(A_4),$$

which can be verified by recursively computing the principal minors. Also, by the elementary operations of rows, we have

$$det = \begin{pmatrix} A_1 & A_2 \\ 0 & I \end{pmatrix} = det \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix} = det(A_1).$$

Finally note that when A and B are square, we have that $det(AB) = det(A)det(B)$. Thus we have

$$det(A) = det(A_1)det(A_4).$$

b) Assume A_1^{-1} and A_4^{-1} exist. Then

$$AA^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

which yields four matrix equations:

1. $A_1B_1 + A_2B_3 = I$,
2. $A_1B_2 + A_2B_4 = 0$,
3. $A_4B_3 = 0$,
4. $A_4B_4 = I$.

From Eqn (4), $B_4 = A_4^{-1}$, with which Eqn (2) yields $B_2 = -A_1^{-1}A_2A_4^{-1}$. Also, from Eqn (3) $B_3 = 0$, with which from Eqn (1) $B_1 = A_1^{-1}$. Therefore,

$$A^{-1} = \begin{pmatrix} A_1^{-1} & -A_1^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{pmatrix}.$$

Exercise 1.2 a)

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix}$$

b) Let us find

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

such that

$$BA = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 - A_3A_1^{-1}A_2 \end{pmatrix}$$

The above equation implies four equations for submatrices

1. $B_1A_1 + B_2A_3 = A_1$,
2. $B_1A_2 + B_2A_4 = A_2$,
3. $B_3A_1 + B_4A_3 = 0$,
4. $B_3A_2 + B_4A_4 = A_4 - A_3A_1^{-1}A_2$.

First two equations yield $B_1 = I$ and $B_2 = 0$. Express B_3 from the third equation as $B_3 = -B_4A_3A_1^{-1}$ and plug it into the fourth. After gathering the terms we get $B_4\left(A_4 - A_3A_1^{-1}A_2\right) = A_4 - A_3A_1^{-1}A_2$, which turns into identity if we set $B_4 = I$. Therefore

$$B = \begin{pmatrix} I & 0 \\ -A_3A_1^{-1} & I \end{pmatrix}$$

c) Using linear operations on rows we see that $det\,(B) = 1$. Then, $det(A) = det(B)det(A) = det\,(BA) = det\,(A_1)\,det\left(A_4 - A_3A_1^{-1}A_2\right)$. Note that $\left(A_4 - A_3A_1^{-1}A_2\right)$ does not have to be invertible for the proof.

Exercise 1.3 We have to prove that $det(I - AB) = det(I - BA)$.
Proof: Since I and $I - BA$ are square,

$$\begin{aligned} det(I - BA) &= det\begin{pmatrix} I & 0 \\ B & I - BA \end{pmatrix} \\ &= det\left(\begin{pmatrix} I & A \\ B & I \end{pmatrix}\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}\right) \\ &= det\begin{pmatrix} I & A \\ B & I \end{pmatrix} det\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}, \end{aligned}$$

yet, from Exercise 1.1, we have

$$det\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} = det(I)det(I) = 1.$$

Thus,

$$det(I - BA) = det\begin{pmatrix} I & A \\ B & I \end{pmatrix}.$$

Now,

$$det\begin{pmatrix} I & A \\ B & I \end{pmatrix} = det\begin{pmatrix} I-AB & 0 \\ B & I \end{pmatrix} = det(I-AB).$$

Therefore

$$det(I-BA) = det(I-AB).$$

Note that $(I-BA)$ is a $q \times q$ matrix while $(I-AB)$ is a $p \times p$ matrix. Thus, when one wants to compute the determinant of $(I-AB)$ or $(I-BA)$, s/he can compare p and q to pick the product (AB or BA) with the smaller size.

b) We have to show that $(I-AB)^{-1}A = A(I-BA)^{-1}$.
Proof: Assume that $(I-BA)^{-1}$ and $(I-AB)^{-1}$ exist. Then,

$$\begin{aligned} A &= A \cdot I = A(I-BA)(I-BA)^{-1} \\ &= (A-ABA)(I-BA)^{-1} \\ &= (I-AB)A(I-BA)^{-1} \\ \rightarrow (I-AB)^{-1}A &= A(I-BA)^{-1}. \end{aligned}$$

This completes the proof.

Exercise 1.6 a) The safest way to find the (element-wise) derivative is by its definition in terms of limits, i.e.

$$\frac{d}{dt}\left(A(t)B(t)\right) = \lim_{\Delta t \to 0} \frac{A(t+\Delta t)B(t+\Delta t) - A(t)B(t)}{\Delta t}$$

We substitute first order Taylor series expansions

$$A(t+\Delta t) = A(t) + \Delta t \frac{d}{dt}A(t) + o(\Delta t)$$

$$B(t+\Delta t) = B(t) + \Delta t \frac{d}{dt}B(t) + o(\Delta t)$$

to obtain

$$\frac{d}{dt}\left(A(t)B(t)\right) = \frac{1}{\Delta t}\left[A(t)B(t) + \Delta t \frac{d}{dt}A(t)B(t) + \Delta t A(t)\frac{d}{dt}B(t) + \text{h.o.t.} - A(t)B(t)\right].$$

Here "h.o.t." stands for the terms

$$\text{h.o.t.} = \left[A(t) + \Delta t \frac{d}{dt}A(t)\right] o(\Delta t) + o(\Delta t)\left[B(t) + \Delta t \frac{d}{dt}B(t)\right] + o(\Delta t^2),$$

a *matrix* quantity, where $\lim_{\Delta t \to 0} \text{h.o.t.}/\Delta t = \mathbf{0}$ (verify). Reducing the expression and taking the limit, we obtain

$$\frac{d}{dt}\left[A(t)B(t)\right] = \frac{d}{dt}A(t)B(t) + A(t)\frac{d}{dt}B(t).$$

b) For this part we write the identity $A^{-1}(t)A(t) = I$. Taking the derivative on both sides, we have

$$\frac{d}{dt}\left[A^{-1}(t)A(t)\right] = \frac{d}{dt}A^{-1}(t)A(t) + A^{-1}(t)\frac{d}{dt}A(t) = \mathbf{0}$$

Rearranging and multiplying on the right by $A^{-1}(t)$, we obtain

$$\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\frac{d}{dt}A(t)A^{-1}(t).$$

Exercise 1.8 Let $X = \{g(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_M x^M \mid \alpha_i \in \mathbb{C}\}$.
a) We have to show that the set $B = \{1, x, \cdots, x^M\}$ is a basis for X.
Proof :

1. First, let's show that elements in B are linearly independent. It is clear that each element in B can not be written as a linear combination of each other. More formally,

$$c_1(1) + c_1(x) + \cdots + c_M(x^M) = 0 \leftrightarrow \forall i \; c_i = 0.$$

   Thus, elements of B are linearly independent.

2. Then, let's show that elements in B span the space X. Every polynomial of order less than or equal to M looks like

$$p(x) = \sum_{i=0}^{M} \alpha_i x^i$$

   for some set of α_i's.
   Therefore, $\{1, x_1, \cdots, x^M\}$ span X.

b) $T : X \to X$ and $T(g(x)) = \frac{d}{dx}g(x)$.

1. Show that T is linear.
   Proof:

$$\begin{aligned} T(ag_1(x) + bg_2(x)) &= \frac{d}{dx}(ag_1(x) + bg_2(x)) \\ &= a\frac{d}{dx}g_1 + b\frac{d}{dx}g_2 \\ &= aT(g_1) + bT(g_2). \end{aligned}$$

   Thus, T is linear.

2. $g(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_M x^M$, so

$$T(g(x)) = \alpha_1 + 2\alpha_2 x + \cdots + M\alpha_M x^{M-1}.$$

   Thus it can be written as follows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & M \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_M \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 2\alpha_2 \\ 3\alpha_3 \\ \vdots \\ M\alpha_M \\ 0 \end{pmatrix}.$$

The big matrix, M, is a matrix representation of T with respect to basis B. The column vector in the left is a representation of $g(x)$ with respect to B. The column vector in the right is $T(g)$ with respect to basis B.

3. Since the matrix M is upper triangular with zeros along diagonal (in fact M is Hessenberg), the eigenvalues are all 0;
$$\lambda_i = 0 \ \forall i = 1, \cdots, M+1.$$

4. One eigenvector of M for $\lambda_1 = 0$ must satisfy $MV_1 = \lambda_1 V_1 = 0$
$$V_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
is one eigenvector. Since λ_i 's are not distinct, the eigenvectors are not necessarily independent. Thus in order to computer the M others, ones uses the generalized eigenvector formula.