6.231 DYNAMIC PROGRAMMING

LECTURE 5

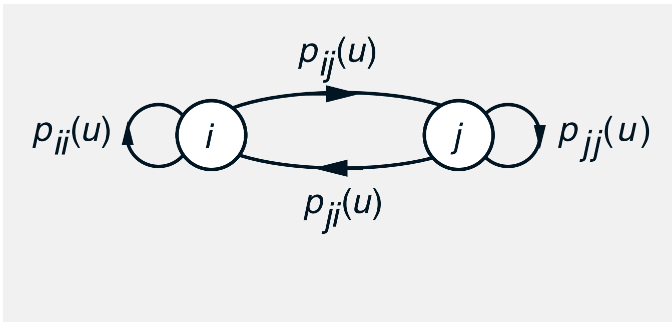
LECTURE OUTLINE

- Review of approximate PI based on projected Bellman equations
- Issues of policy improvement
  - Exploration enhancement in policy evaluation
  - Oscillations in approximate PI
- Aggregation – An alternative to the projected equation/Galerkin approach
- Examples of aggregation
- Simulation-based aggregation
- Relation between aggregation and projected equations

REVIEW

DISCOUNTED MDP

- System: Controlled Markov chain with states $i = 1, \ldots, n$ and finite set of controls $u \in U(i)$
- Transition probabilities: $p_{ij}(u)$



- Cost of a policy $\pi = \{\mu_0, \mu_1, \ldots\}$ starting at state i:

$$J_\pi(i) = \lim_{N \to \infty} E\left\{ \sum_{k=0}^{N} \alpha^k g\big(i_k, \mu_k(i_k), i_{k+1}\big) \mid i = i_0 \right\}$$

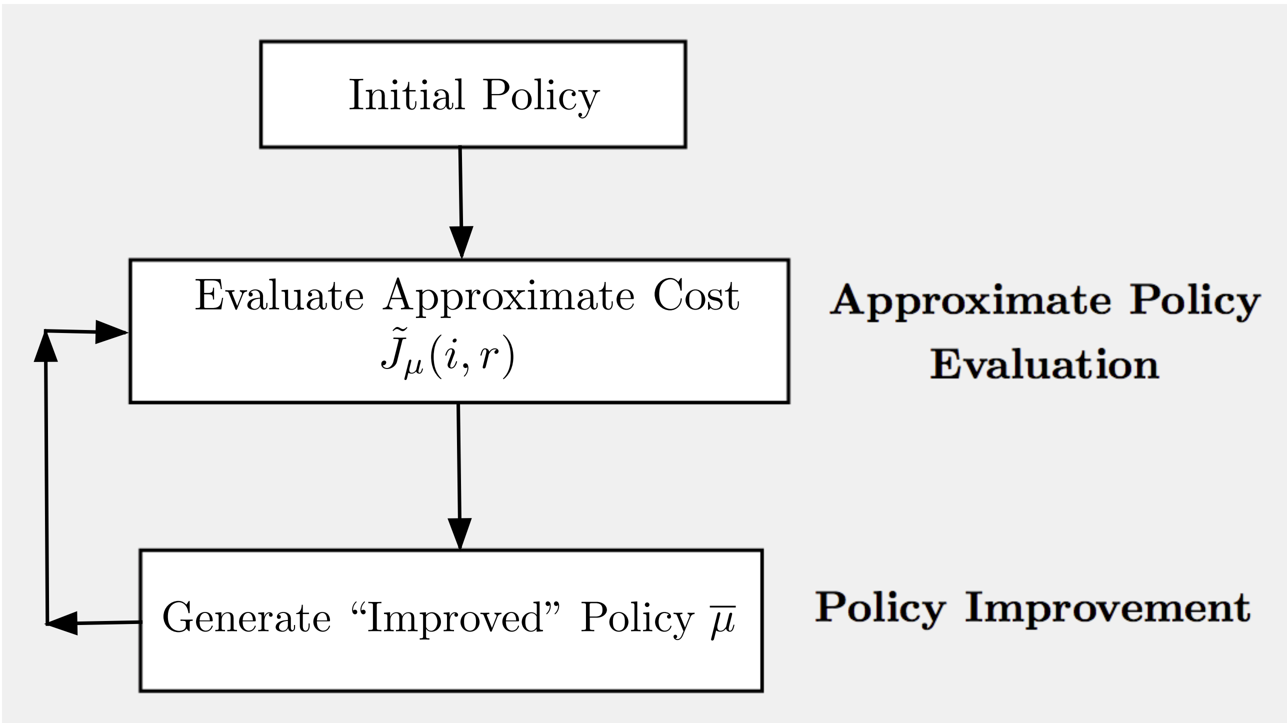
with $\alpha \in [0, 1)$

- Shorthand notation for DP mappings

$$(TJ)(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i, u, j) + \alpha J(j)\big), \quad i = 1, \ldots, n,$$

$$(T_\mu J)(i) = \sum_{j=1}^{n} p_{ij}\big(\mu(i)\big)\Big(g\big(i, \mu(i), j\big) + \alpha J(j)\Big), \quad i = 1, \ldots, n$$

APPROXIMATE PI



- Evaluation of typical policy μ: Linear cost function approximation

$$\tilde{J}_\mu(r) = \Phi r$$

where Φ is full rank $n \times s$ matrix with columns the basis functions, and ith row denoted $\phi(i)'$.

- Policy "improvement" to generate $\overline{\mu}$:

$$\overline{\mu}(i) = \arg \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u) \big(g(i, u, j) + \alpha \phi(j)' r \big)$$

EVALUATION BY PROJECTED EQUATIONS

- Approximate policy evaluation by solving

$$\Phi r = \Pi T_\mu(\Phi r)$$

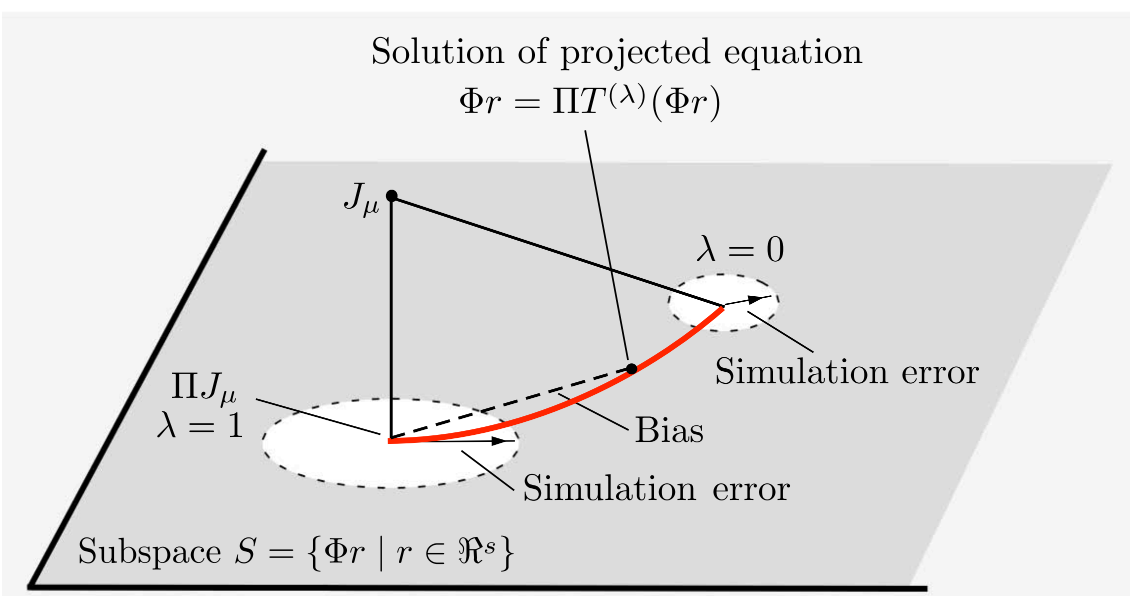
Π: weighted Euclidean projection; special nature of the steady-state distribution weighting.

- Implementation by simulation (single long trajectory using current policy - important to make ΠT_μ a contraction). LSTD, LSPE methods.

- Multistep option: Solve $\Phi r = \Pi T_\mu^{(\lambda)}(\Phi r)$ with

$$T_\mu^{(\lambda)} = (1-\lambda)\sum_{\ell=0}^{\infty} \lambda^\ell T_\mu^{\ell+1}, \qquad 0 \le \lambda < 1$$

  - As $\lambda \uparrow 1$, $\Pi T_\mu^{(\lambda)}$ becomes a contraction for any projection norm (allows changes in Π)
  - Bias-variance tradeoff



ISSUES OF POLICY IMPROVEMENT

EXPLORATION

- **1st major issue:** exploration. To evaluate μ, we need to generate cost samples using μ

- This biases the simulation by underrepresenting states that are unlikely to occur under μ.

- As a result, the cost-to-go estimates of these underrepresented states may be highly inaccurate, and seriously impact the "improved policy" $\overline{\mu}$.

- This is known as inadequate exploration - a particularly acute difficulty when the randomness embodied in the transition probabilities is "relatively small" (e.g., a deterministic system).

- To deal with this we must change the sampling mechanism and modify the simulation formulas.

- Solve

$$\Phi r = \overline{\Pi} T_\mu(\Phi r)$$

where $\overline{\Pi}$ is projection with respect to an exploration-enhanced norm [uses a weight distribution $\zeta = (\zeta_1, \ldots, \zeta_n)$].

- ζ is more "balanced" than ξ the steady-state distribution of the Markov chain of μ.

- This also addresses any lack of ergodicity of μ.

EXPLORATION MECHANISMS

- One possibility: Use multiple short simulation trajectories instead of single long trajectory starting from a rich mixture of states. This is known as geometric sampling, or free-form sampling.
  - By properly choosing the starting states, we enhance exploration
  - The simulation formulas for LSTD(λ) and LSPE(λ) have to be modified to yield the solution of $\Phi r = \overline{\Pi} T_\mu^{(\lambda)}(\Phi r)$ (see the DP text)
- Another possibility: Use a modified policy to generate a single long trajectory. This is called an off-policy approach.
  - Modify the transition probabilities of μ to enhance exploration
  - Again the simulation formulas for LSTD(λ) and LSPE(λ) have to be modified to yield the solution of $\Phi r = \overline{\Pi} T_\mu^{(\lambda)}(\Phi r)$ (use of importance sampling; see the DP text)
- With larger values of $\lambda > 0$ the contraction property of $\overline{\Pi} T_\mu^{(\lambda)}$ is maintained.
- LSTD may be used without $\overline{\Pi} T_\mu^{(\lambda)}$ being a contraction ... LSPE and TD require a contraction.

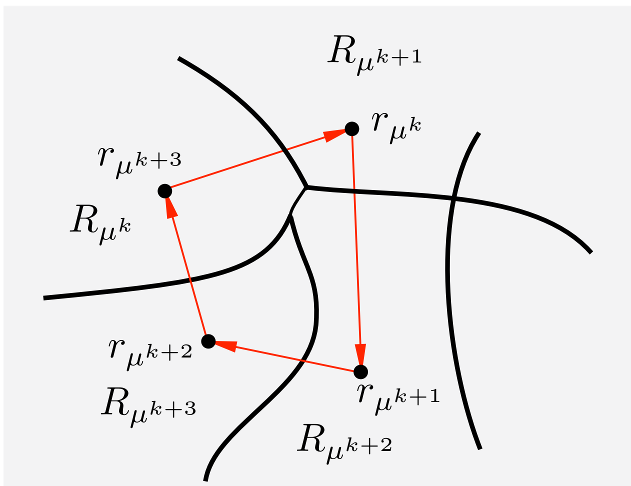
POLICY ITERATION ISSUES: OSCILLATIONS

- **2nd major issue:** oscillation of policies

- Analysis using the greedy partition of the space of weights r: R_μ is the set of parameter vectors r for which μ is greedy with respect to $\tilde{J}(\cdot; r) = \Phi r$

$$R_\mu = \big\{ r \mid T_\mu(\Phi r) = T(\Phi r) \big\} \qquad \forall\ \mu$$

If we use r in R_μ the next "improved" policy is μ



- If policy evaluation is exact, there is a finite number of possible vectors r_μ, (one per μ)

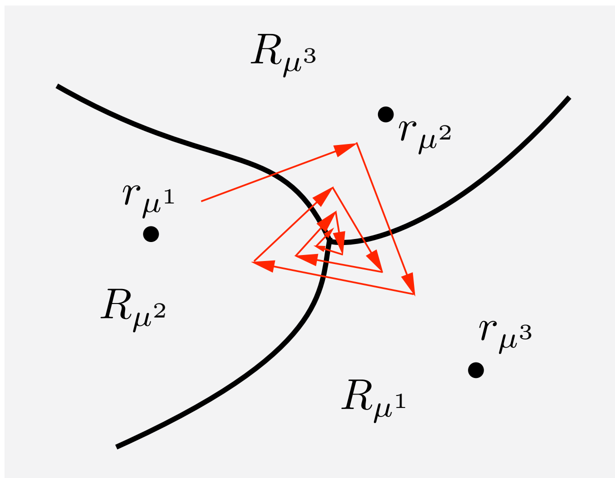
- The algorithm ends up repeating some cycle of policies $\mu^k, \mu^{k+1}, \ldots, \mu^{k+m}$ with

$$r_{\mu^k} \in R_{\mu^{k+1}},\ r_{\mu^{k+1}} \in R_{\mu^{k+2}}, \ldots, r_{\mu^{k+m}} \in R_{\mu^k}$$

- Many different cycles are possible

MORE ON OSCILLATIONS/CHATTERING

- In the case of optimistic policy iteration a different picture holds (policy evaluation does not produce exactly r_μ)



- Oscillations of weight vector r are less violent, but the "limit" point is meaningless!

- Fundamentally, oscillations are due to the lack of monotonicity of the projection operator, i.e., $J \leq J'$ does not imply $\Pi J \leq \Pi J'$.

- If approximate PI uses an evaluation of the form

$$\Phi r = (W T_\mu)(\Phi r)$$

with W: monotone and $W T_\mu$: contraction, the policies converge (to a possibly nonoptimal limit).

- These conditions hold when aggregation is used

AGGREGATION

PROBLEM APPROXIMATION - AGGREGATION

- Another major idea in ADP is to approximate J^* or J_μ with the cost-to-go functions of a simpler problem.

- Aggregation is a systematic approach for problem approximation. Main elements:
  - Introduce a few "aggregate" states, viewed as the states of an "aggregate" system
  - Define transition probabilities and costs of the aggregate system, by relating original system states with aggregate states
  - Solve (exactly or approximately) the "aggregate" problem by any kind of VI or PI method (including simulation-based methods)

- If $\hat{R}(y)$ is the optimal cost of aggregate state y, we use the approximation

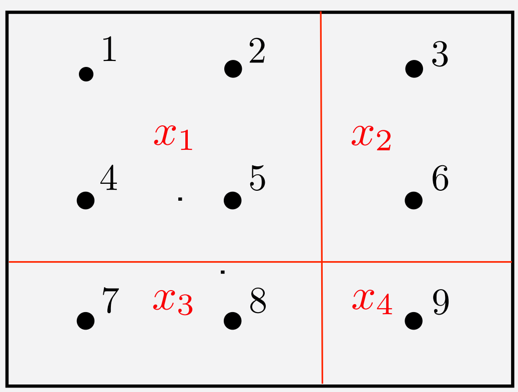
$$J^*(j) \approx \sum_y \phi_{jy} \hat{R}(y), \qquad \forall\, j$$

where ϕ_{jy} are the aggregation probabilities, encoding the "degree of membership of j in the aggregate state y"

- This is a linear architecture: ϕ_{jy} are the features of state j

HARD AGGREGATION EXAMPLE

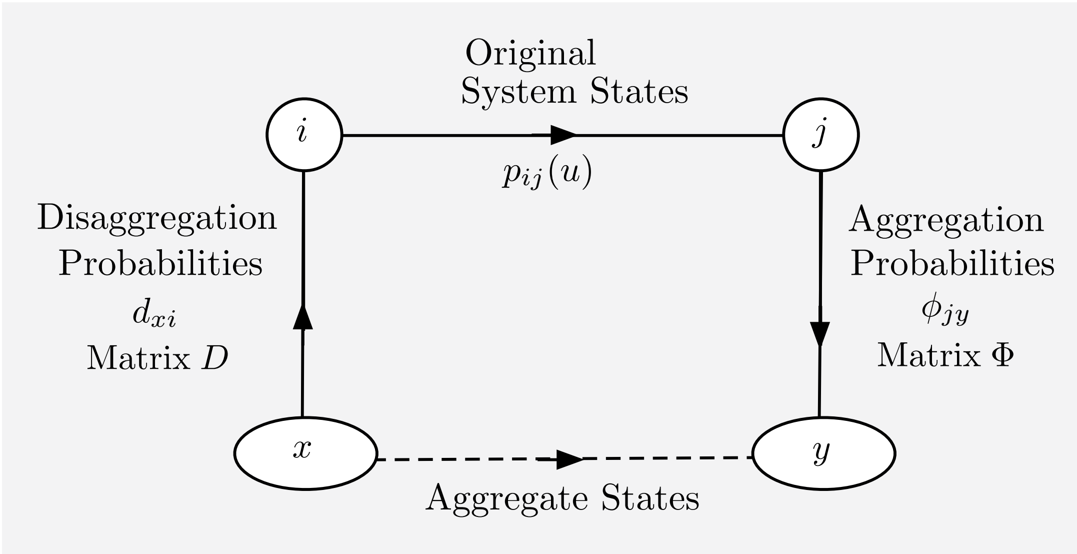
- Group the original system states into subsets, and view each subset as an aggregate state
- Aggregation probs.: $\phi_{jy} = 1$ if j belongs to aggregate state y (piecewise constant approx).



$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- What should be the "aggregate" transition probs. out of x?
- Select $i \in x$ and use the transition probs. of i. But which i should I use?
- The simplest possibility is to assume that all states i in x are equally likely.
- A generalization is to randomize, i.e., use "disaggregation probabilities" d_{xi}: Roughly, the "degree to which i is representative of x."

AGGREGATION/DISAGGREGATION PROBS



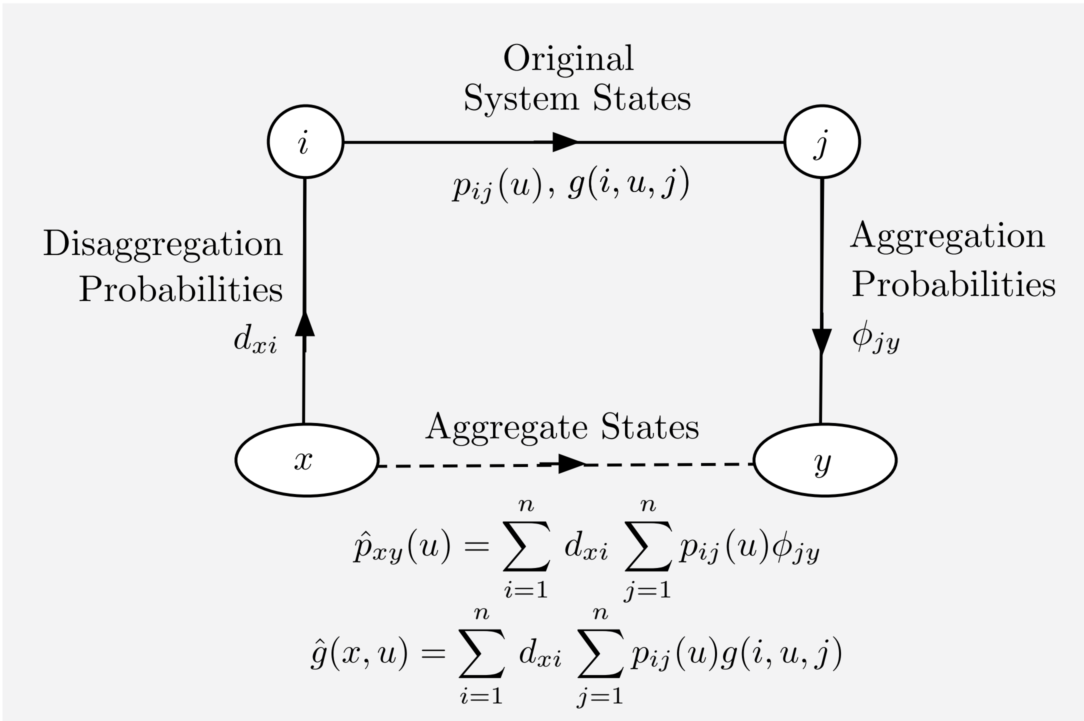
- Define the aggregate system transition probabilities via two (somewhat arbitrary) choices.

- For each original system state j and aggregate state y, the aggregation probability ϕ_{jy}
  - Roughly, the "degree of membership of j in the aggregate state y."
  - In hard aggregation, $\phi_{jy} = 1$ if state j belongs to aggregate state/subset y.

- For each aggregate state x and original system state i, the disaggregation probability d_{xi}
  - Roughly, the "degree to which i is representative of x."

- Aggregation scheme is defined by the two matrices D and Φ. The rows of D and Φ must be probability distributions.

AGGREGATE SYSTEM DESCRIPTION



- The transition probability from aggregate state x to aggregate state y under control u

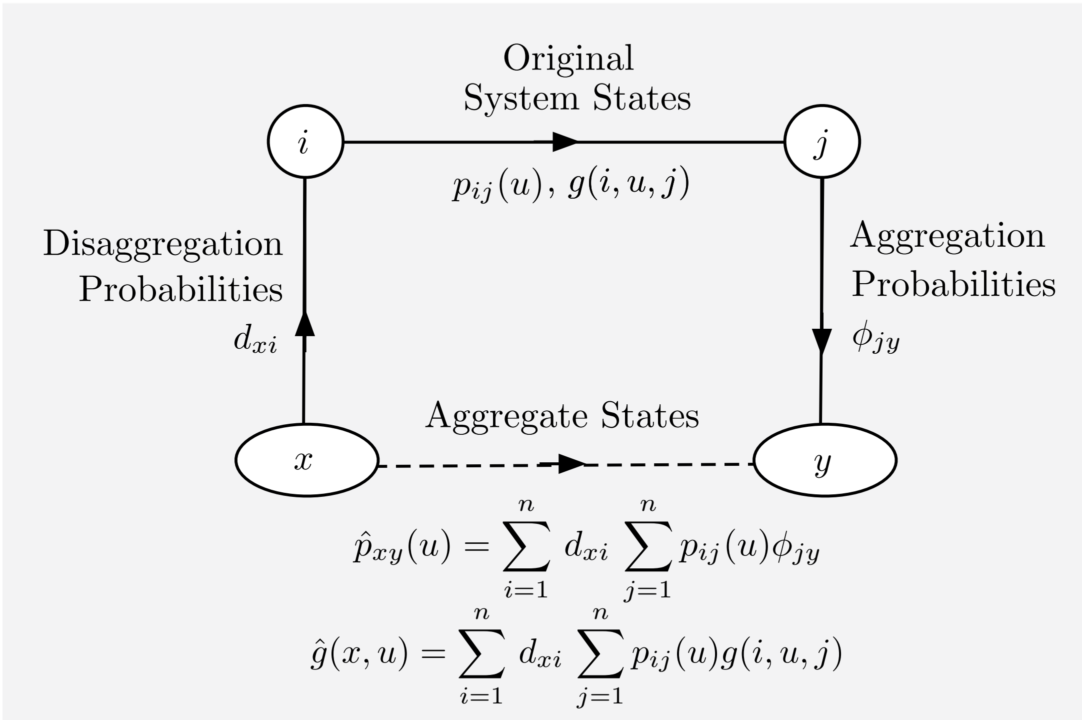
$$\hat{p}_{xy}(u) = \sum_{i=1}^{n} d_{xi} \sum_{j=1}^{n} p_{ij}(u)\phi_{jy}, \quad \text{or } \hat{P}(u) = DP(u)\Phi$$

where the rows of D and Φ are the disaggregation and aggregation probs.

- The expected transition cost is

$$\hat{g}(x,u) = \sum_{i=1}^{n} d_{xi} \sum_{j=1}^{n} p_{ij}(u)g(i,u,j), \quad \text{or } \hat{g} = DP(u)g$$

AGGREGATE BELLMAN'S EQUATION



- The optimal cost function of the aggregate problem, denoted $\hat R$, is

$$\hat R(x) = \min_{u \in U} \left[\hat g(x,u) + \alpha \sum_y \hat p_{xy}(u) \hat R(y) \right], \qquad \forall\ x$$

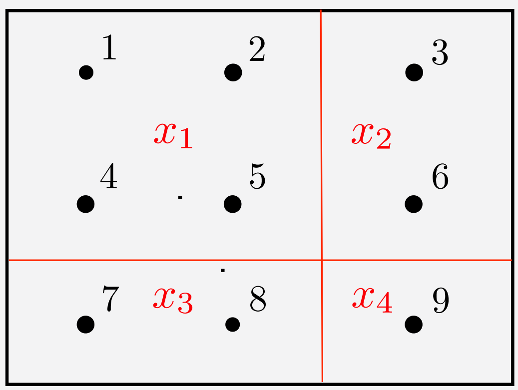
Bellman's equation for the aggregate problem.

- The optimal cost function J^* of the original problem is approximated by $\tilde J$ given by

$$\tilde J(j) = \sum_y \phi_{jy} \hat R(y), \qquad \forall\ j$$

EXAMPLE I: HARD AGGREGATION

- Group the original system states into subsets, and view each subset as an aggregate state
- Aggregation probs.: $\phi_{jy} = 1$ if j belongs to aggregate state y.



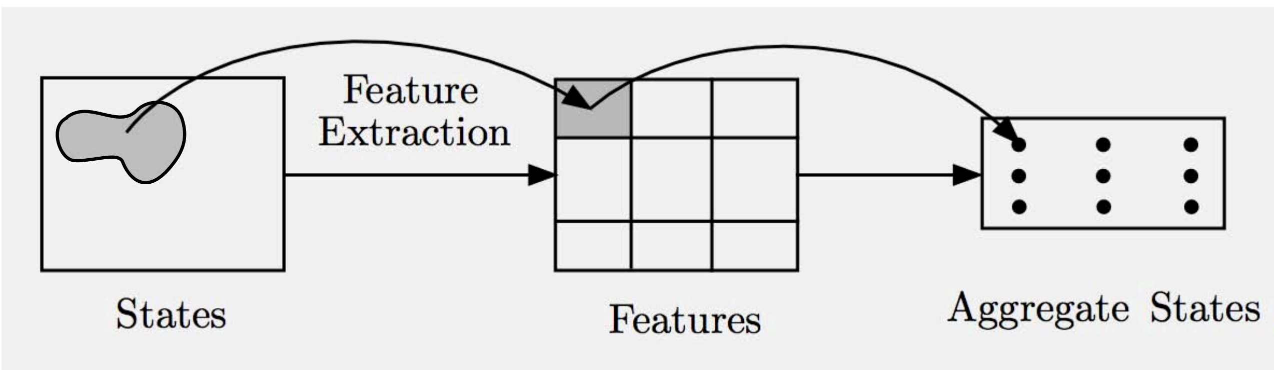
$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Disaggregation probs.: There are many possibilities, e.g., all states i within aggregate state x have equal prob. d_{xi}.
- If optimal cost vector J^* is piecewise constant over the aggregate states/subsets, hard aggregation is exact. Suggests grouping states with "roughly equal" cost into aggregates.
- A variant: Soft aggregation (provides "soft boundaries" between aggregate states).

EXAMPLE II: FEATURE-BASED AGGREGATION

- Important question: How do we group states together?

- If we know good features, it makes sense to group together states that have "similar features"



- A general approach for passing from a feature-based state representation to a hard aggregation-based architecture

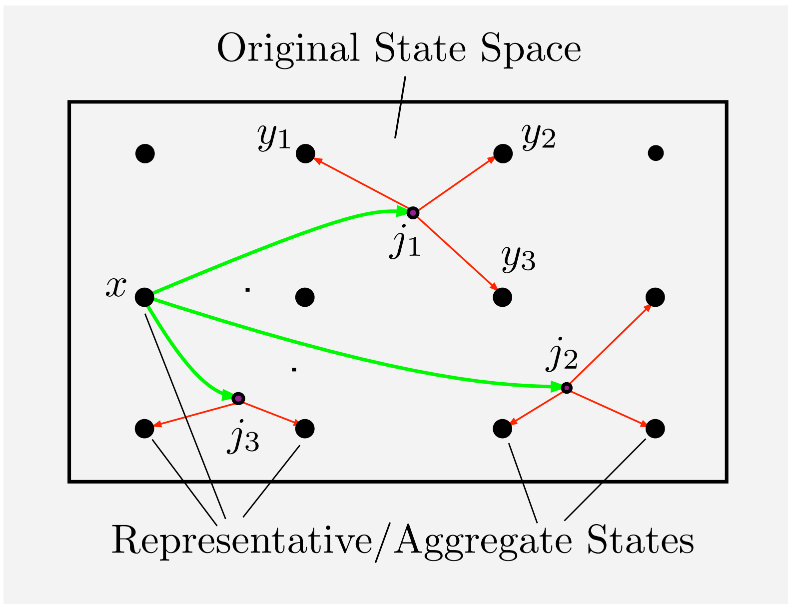
- Essentially discretize the features and generate a corresponding piecewise constant approximation to the optimal cost function

- Aggregation-based architecture is more powerful (it is nonlinear in the features)

- ... but may require many more aggregate states to reach the same level of performance as the corresponding linear feature-based architecture

EXAMPLE III: REP. STATES/COARSE GRID

- Choose a collection of "representative" original system states, and associate each one of them with an aggregate state



- Disaggregation probabilities are $d_{xi} = 1$ if i is equal to representative state x.

- Aggregation probabilities associate original system states with convex combinations of representative states

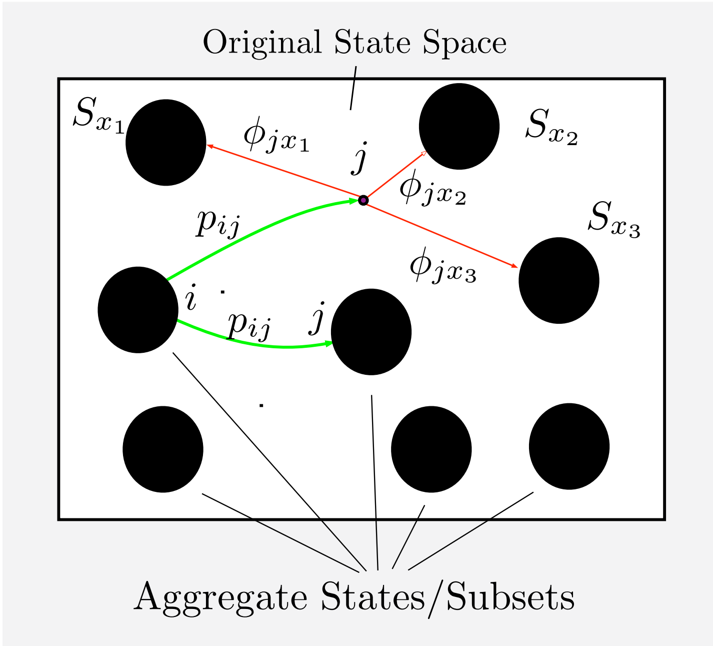
$$j \sim \sum_{y \in \mathcal{A}} \phi_{jy} y$$

- Well-suited for Euclidean space discretization

- Extends nicely to continuous state space, including belief space of POMDP

EXAMPLE IV: REPRESENTATIVE FEATURES

- Here the aggregate states are nonempty subsets of original system states. Common case: Each S_x is a group of states with "similar features"

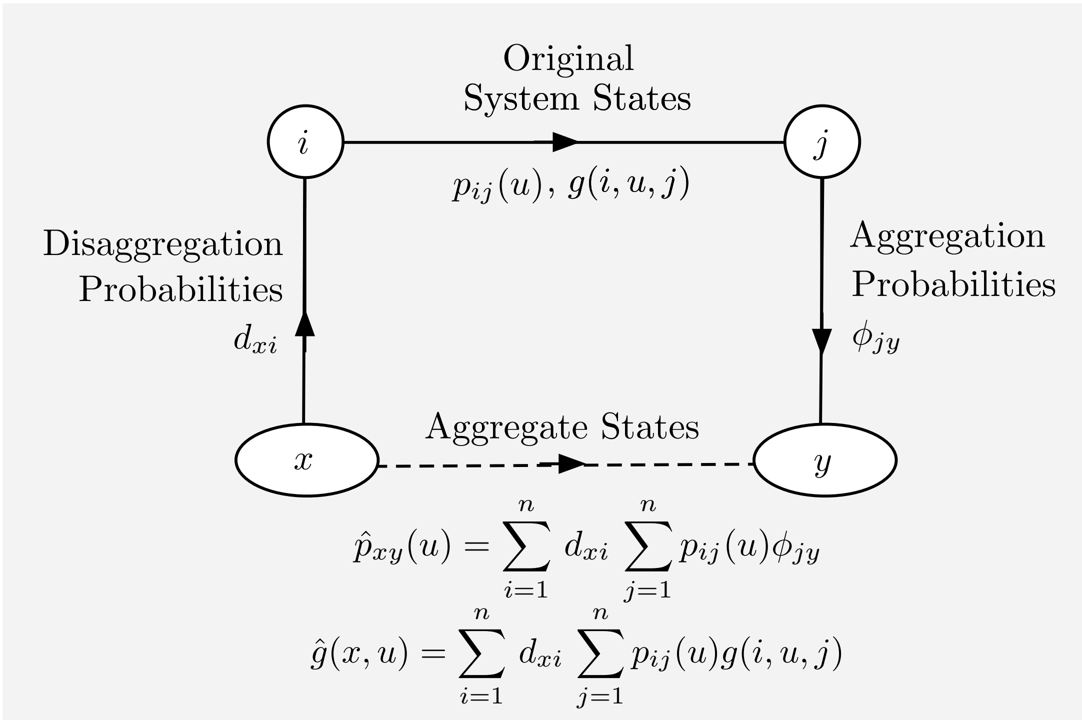


- Restrictions:
  - The aggregate states/subsets are disjoint.
  - The disaggregation probabilities satisfy $d_{xi} > 0$ if and only if $i \in x$.
  - The aggregation probabilities satisfy $\phi_{jy} = 1$ for all $j \in y$.

- Hard aggregation is a special case: $\cup_x S_x = \{1, \ldots, n\}$

- Aggregation with representative states is a special case: S_x consists of just one state

APPROXIMATE PI BY AGGREGATION



- Consider approximate PI for the original problem, with policy evaluation done by aggregation.

- Evaluation of policy μ: $\tilde{J} = \Phi R$, where $R = DT_\mu(\Phi R)$ (R is the vector of costs of aggregate states for μ). Can be done by simulation.

- Looks like projected equation $\Phi R = \Pi T_\mu(\Phi R)$ (but with ΦD in place of Π).

- Advantage: It has no problem with oscillations.

- Disadvantage: The rows of D and Φ must be probability distributions.

ADDITIONAL ISSUES OF AGGREGATION

ALTERNATIVE POLICY ITERATION

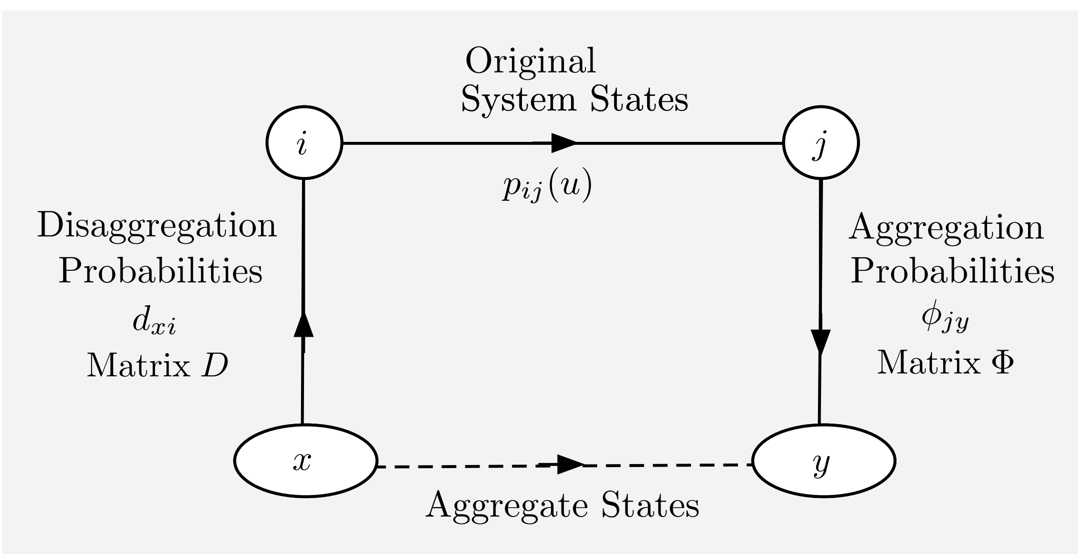
- The preceding PI method uses policies that assign a control to each aggregate state.

- An alternative is to use PI for the combined system, involving the Bellman equations:

$$R^*(x) = \sum_{i=1}^{n} d_{xi} \tilde{J}_0(i), \qquad \forall\, x,$$

$$\tilde{J}_0(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha \tilde{J}_1(j)\big), \quad i = 1, \ldots, n,$$

$$\tilde{J}_1(j) = \sum_{y \in \mathcal{A}} \phi_{jy} R^*(y), \qquad j = 1, \ldots, n.$$



- Simulation-based PI and VI are still possible.

RELATION OF AGGREGATION/PROJECTION

- Compare aggregation and projected equations

$$\Phi R = \Phi D T(\Phi R), \qquad \Phi r = \Pi T(\Phi r)$$

- If ΦD is a projection (with respect to some weighted Euclidean norm), then the methodology of projected equations applies to aggregation

- Hard aggregation case: ΦD can be verified to be projection with respect to weights ξ_i proportional to the disaggregation probabilities d_{xi}

- Aggregation with representative features case: ΦD can be verified to be a semi-norm projection with respect to weights ξ_i proportional to d_{xi}

- A (weighted) Euclidean semi-norm is defined by $\|J\|_\xi = \sqrt{\sum_{i=1}^n \xi_i \big(J(i)\big)^2}$, where $\xi = (\xi_1, \ldots, \xi_n)$, with $\xi_i \geq 0$.

- If $\Phi' \Xi \Phi$ is invertible, the entire theory and algorithms of projected equations generalizes to semi-norm projected equations [including multi-step methods such as LSTD/LSPE/TD(λ)].

- Reference: Yu and Bertsekas, "Weighted Bellman Equations and their Applications in Approximate Dynamic Programming," MIT Report, 2012.

DISTRIBUTED AGGREGATION I

- We consider decomposition/distributed solution of large-scale discounted DP problems by hard aggregation.
- Partition the original system states into subsets $S_1, \ldots, S_m$.
- Distributed VI Scheme: Each subset S_ℓ
  - Maintains detailed/exact local costs

$$J(i) \quad \text{for every original system state } i \in S_\ell$$

    using aggregate costs of other subsets
  - Maintains an aggregate cost $R(\ell) = \sum_{i \in S_\ell} d_{\ell i} J(i)$
  - Sends $R(\ell)$ to other aggregate states
- $J(i)$ and $R(\ell)$ are updated by VI according to

$$J_{k+1}(i) = \min_{u \in U(i)} H_\ell(i, u, J_k, R_k), \qquad \forall\ i \in S_\ell$$

with R_k being the vector of $R(\ell)$ at time k, and

$$H_\ell(i, u, J, R) = \sum_{j=1}^{n} p_{ij}(u) g(i, u, j) + \alpha \sum_{j \in S_\ell} p_{ij}(u) J(j) + \alpha \sum_{j \in S_{\ell'},\, \ell' \neq \ell} p_{ij}(u) R(\ell')$$

DISTRIBUTED AGGREGATION II

- Can show that this iteration involves a sup-norm contraction mapping of modulus α, so it converges to the unique solution of the system of equations in (J, R)

$$J(i) = \min_{u \in U(i)} H_\ell(i, u, J, R), \quad R(\ell) = \sum_{i \in S_\ell} d_{\ell i} J(i),$$

$$\forall\, i \in S_\ell,\ \ell = 1, \ldots, m.$$

- This follows from the fact that $\{d_{\ell i} \mid i = 1, \ldots, n\}$ is a probability distribution.

- View these equations as a set of Bellman equations for an "aggregate" DP problem. The difference is that the mapping H involves $J(j)$ rather than $R\big(x(j)\big)$ for $j \in S_\ell$.

- In an asynchronous version of the method, the aggregate costs $R(\ell)$ may be outdated to account for communication "delays" between aggregate states.

- Convergence can be shown using the general theory of asynchronous distributed computation, briefly described in the 2nd lecture (see the text).

6.231 Dynamic Programming and Stochastic Control
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