6.231 DYNAMIC PROGRAMMING

LECTURE 19

LECTURE OUTLINE

- We begin a lecture series on approximate DP.
- Reading: Chapters 6 and 7, Vol. 2 of the text.
- Today we discuss some general issues about approximation and simulation
- We classify/overview the main approaches:
  - Approximation in policy space (policy parametrization, gradient methods, random search)
  - Approximation in value space (approximate PI, approximate VI, Q-Learning, Bellman error approach, approximate LP)
  - Rollout/Simulation-based single policy iteration (will not discuss this further)
  - Approximation in value space using problem approximation (simplification - forms of aggregation - limited lookahead) - will not discuss much

GENERAL ORIENTATION TO ADP

- ADP (late 80s - present) is a breakthrough methodology that allows the application of DP to problems with many or infinite number of states.
- Other names for ADP are:
  - "reinforcement learning" (RL)
  - "neuro-dynamic programming" (NDP)
- We will mainly adopt an n-state discounted model (the easiest case - but think of HUGE n).
- Extensions to other DP models (continuous space, continuous-time, not discounted) are possible (but more quirky). We will set aside for later.
- There are many approaches:
  - Problem approximation and 1-step lookahead
  - Simulation-based approaches (we will focus on these)
- Simulation-based methods are of three types:
  - Rollout (we will not discuss further)
  - Approximation in policy space
  - Approximation in value space

WHY DO WE USE SIMULATION?

- One reason: Computational complexity advantage in computing expected values and sums/inner products involving a very large number of terms
  - Speeds up linear algebra: Any sum $\sum_{i=1}^{n} a_i$ can be written as an expected value

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \xi_i \frac{a_i}{\xi_i} = E_\xi \left\{ \frac{a_i}{\xi_i} \right\},$$

    where ξ is any prob. distribution over $\{1, \ldots, n\}$
  - It is approximated by generating many samples $\{i_1, \ldots, i_k\}$ from $\{1, \ldots, n\}$, according to ξ, and Monte Carlo averaging:

$$\sum_{i=1}^{n} a_i = E_\xi \left\{ \frac{a_i}{\xi_i} \right\} \approx \frac{1}{k} \sum_{t=1}^{k} \frac{a_{i_t}}{\xi_{i_t}}$$

  - Choice of ξ makes a difference. Importance sampling methodology.

- Simulation is also convenient when an analytical model of the system is unavailable, but a simulation/computer model is possible.

APPROXIMATION IN POLICY SPACE

- A brief discussion; we will return to it later.
- Use parametrization $\mu(i;r)$ of policies with a vector $r = (r_1, \ldots, r_s)$. Examples:
  - Polynomial, e.g., $\mu(i;r) = r_1 + r_2 \cdot i + r_3 \cdot i^2$
  - Multi-warehouse inventory system: $\mu(i;r)$ is threshold policy with thresholds $r = (r_1, \ldots, r_s)$
- Optimize the cost over r. For example:
  - Each value of r defines a stationary policy, with cost starting at state i denoted by $\tilde{J}(i;r)$.
  - Let $(p_1, \ldots, p_n)$ be some probability distribution over the states, and minimize over r

$$\sum_{i=1}^{n} p_i \tilde{J}(i;r)$$

  - Use a random search, gradient, or other method
- A special case: The parameterization of the policies is indirect, through a cost approximation architecture $\hat{J}$, i.e.,

$$\mu(i;r) \in \arg\min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha \hat{J}(j;r)\big)$$

APPROXIMATION IN VALUE SPACE

- Approximate J^* or J_μ from a parametric class $\tilde{J}(i;r)$ where i is the current state and $r = (r_1, \ldots, r_m)$ is a vector of "tunable" scalars weights

- Use $\tilde{J}$ in place of J^* or J_μ in various algorithms and computations (VI, PI, LP)

- Role of r: By adjusting r we can change the "shape" of $\tilde{J}$ so that it is "close" to J^* or J_μ

- Two key issues:
  - The choice of parametric class $\tilde{J}(i;r)$ (the approximation architecture)
  - Method for tuning the weights ("training" the architecture)

- Success depends strongly on how these issues are handled ... also on insight about the problem

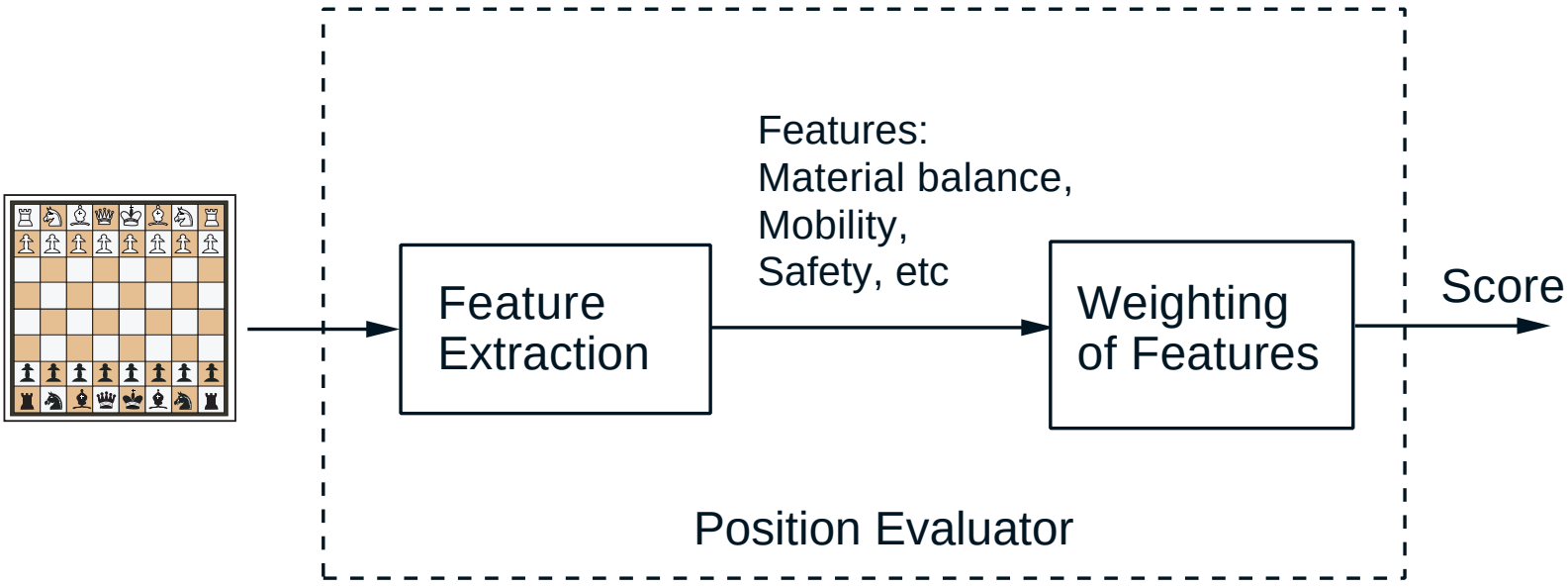
- A simulator may be used, particularly when there is no mathematical model of the system

- We will focus on simulation, but this is not the only possibility

- We may also use parametric approximation for Q-factors

APPROXIMATION ARCHITECTURES

- Divided in linear and nonlinear [i.e., linear or nonlinear dependence of $\tilde{J}(i;r)$ on r]
- Linear architectures are easier to train, but nonlinear ones (e.g., neural networks) are richer
- Computer chess example:
  - Think of board position as state and move as control
  - Uses a feature-based position evaluator that assigns a score (or approximate Q-factor) to each position/move



- Relatively few special features and weights, and multistep lookahead

LINEAR APPROXIMATION ARCHITECTURES

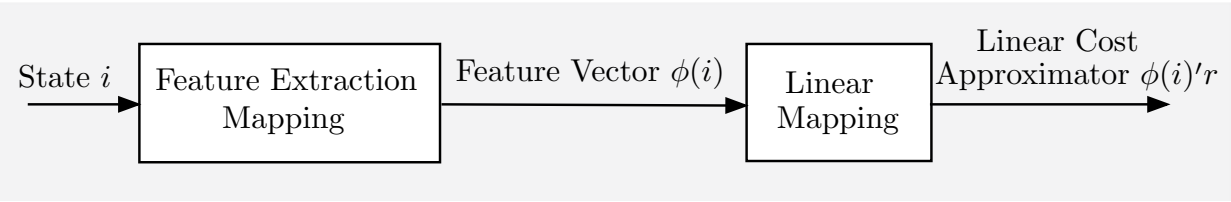
- Often, the features encode much of the nonlinearity inherent in the cost function approximated

- Then the approximation may be quite accurate without a complicated architecture. (Extreme example: The ideal feature is the true cost function)

- With well-chosen features, we can use a linear architecture:

$$\tilde{J}(i;r) = \phi(i)'r, \ \forall \ i \quad \text{or} \quad \tilde{J}(r) = \Phi r = \sum_{j=1}^{s} \Phi_j r_j$$

Φ: the matrix whose rows are $\phi(i)'$, $i = 1, \ldots, n$, Φ_j is the jth column of Φ



- This is approximation on the subspace

$$S = \{\Phi r \mid r \in \Re^s\}$$

spanned by the columns of Φ (basis functions)

- Many examples of feature types: Polynomial approximation, radial basis functions, domain specific, etc

ILLUSTRATIONS: POLYNOMIAL TYPE

• Polynomial Approximation, e.g., a quadratic approximating function. Let the state be $i = (i_1, \ldots, i_q)$ (i.e., have q "dimensions") and define

$$\phi_0(i) = 1, \quad \phi_k(i) = i_k, \quad \phi_{km}(i) = i_k i_m, \quad k, m = 1, \ldots, q$$

Linear approximation architecture:

$$\tilde{J}(i; r) = r_0 + \sum_{k=1}^{q} r_k i_k + \sum_{k=1}^{q} \sum_{m=k}^{q} r_{km} i_k i_m,$$

where r has components r_0, r_k, and r_{km}.

• Interpolation: A subset I of special/representative states is selected, and the parameter vector r has one component r_i per state $i \in I$. The approximating function is

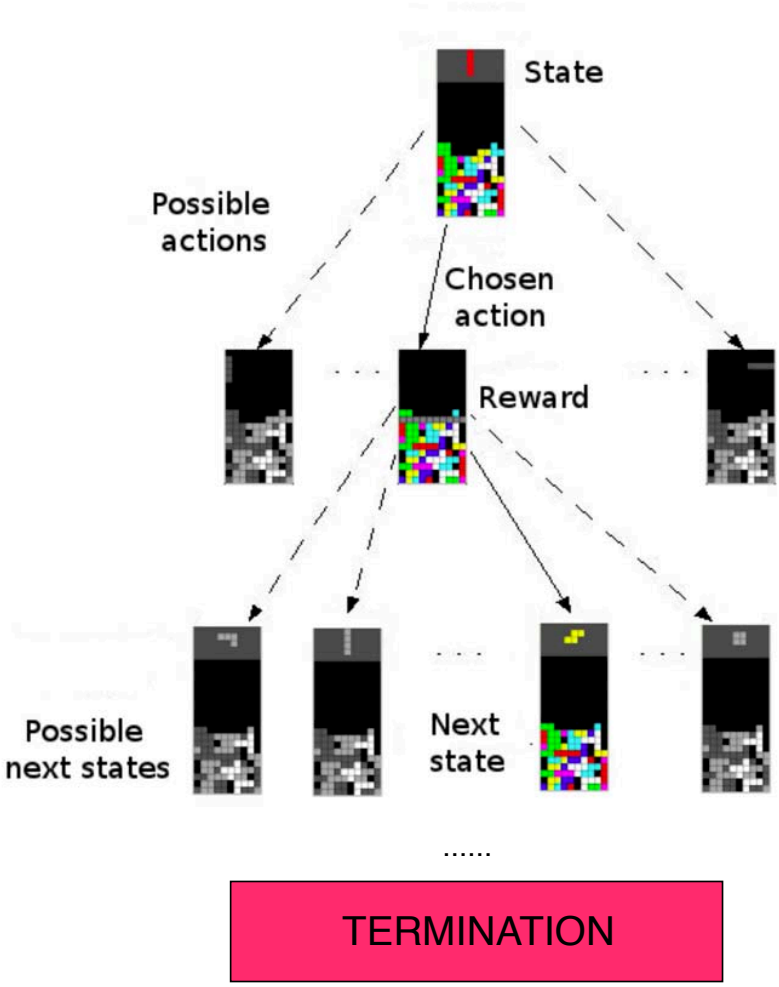
$$\tilde{J}(i; r) = r_i, \qquad i \in I,$$

$\tilde{J}(i; r) =$ interpolation using the values at $i \in I$, $i \notin I$

For example, piecewise constant, piecewise linear, more general polynomial interpolations.

A DOMAIN SPECIFIC EXAMPLE

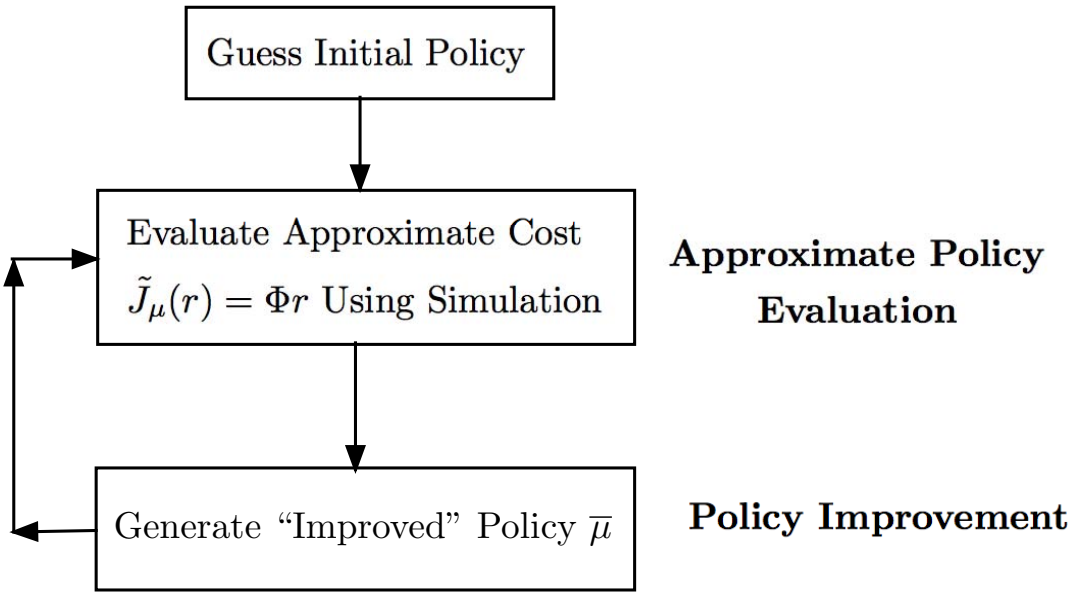
- Tetris game (used as testbed in competitions)



- $J^*(i)$: optimal score starting from position i
- Number of states $> 2^{200}$ (for 10×20 board)
- Success with just 22 features, readily recognized by tetris players as capturing important aspects of the board position (heights of columns, etc)

APPROX. PI - OPTION TO APPROX. J_μ OR Q_μ

- Use simulation to approximate the cost J_μ of the current policy μ
- Generate "improved" policy $\overline{\mu}$ by minimizing in (approx.) Bellman equation



- Altenatively approximate the Q-factors of μ
- A survey reference: D. P. Bertsekas, "Approximate Policy Iteration: A Survey and Some New Methods," J. of Control Theory and Appl., Vol. 9, 2011, pp. 310-335.

DIRECTLY APPROXIMATING J^* OR Q^*

- Approximation of the optimal cost function J^* directly (without PI)
  - Q-Learning: Use a simulation algorithm to approximate the Q-factors

$$Q^*(i,u) = g(i,u) + \alpha \sum_{j=1}^{n} p_{ij}(u) J^*(j);$$

    and the optimal costs

$$J^*(i) = \min_{u \in U(i)} Q^*(i,u)$$

  - Bellman Error approach: Find r to

$$\min_r E_i \left\{ \left(\tilde{J}(i;r) - (T\tilde{J})(i;r) \right)^2 \right\}$$

    where $E_i\{\cdot\}$ is taken with respect to some distribution over the states
  - Approximate Linear Programming (we will not discuss here)

- Q-learning can also be used with approximations

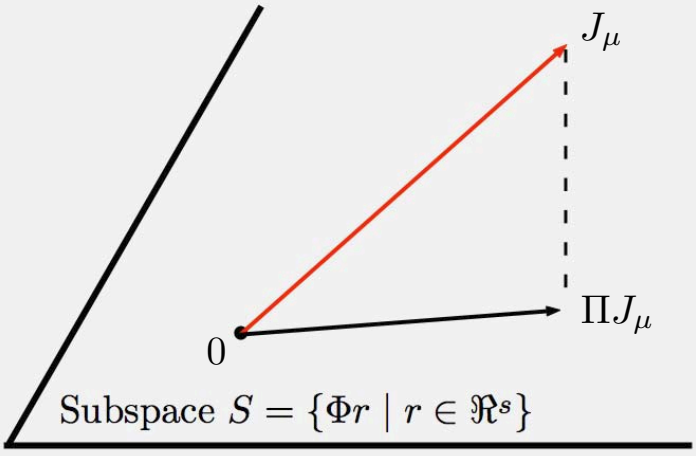
- Q-learning and Bellman error approach can also be used for policy evaluation

DIRECT POLICY EVALUATION

- Can be combined with regular and optimistic policy iteration
- Find r that minimizes $\|J_\mu - \tilde{J}(\cdot, r)\|_\xi^2$, i.e.,

$$\sum_{i=1}^{n} \xi_i \big(J_\mu(i) - \tilde{J}(i,r)\big)^2, \qquad \xi_i\text{: some pos. weights}$$

- Nonlinear architectures may be used
- The linear architecture case: Amounts to projection of J_μ onto the approximation subspace



Direct Method: Projection of cost vector J_μ

- Solution by linear least squares methods

POLICY EVALUATION BY SIMULATION

- Projection by Monte Carlo Simulation: Compute the projection ΠJ_μ of J_μ on subspace $S = \{\Phi r \mid r \in \Re^s\}$, with respect to a weighted Euclidean norm $\|\cdot\|_\xi$
- Equivalently, find Φr^*, where

$$r^* = \arg\min_{r\in\Re^s} \|\Phi r - J_\mu\|_\xi^2 = \arg\min_{r\in\Re^s} \sum_{i=1}^n \xi_i \big(J_\mu(i) - \phi(i)'r\big)^2$$

- Setting to 0 the gradient at r^*,

$$r^* = \left(\sum_{i=1}^n \xi_i \phi(i)\phi(i)'\right)^{-1} \sum_{i=1}^n \xi_i \phi(i) J_\mu(i)$$

- Generate samples $\big\{(i_1, J_\mu(i_1)), \ldots, (i_k, J_\mu(i_k))\big\}$ using distribution ξ
- Approximate by Monte Carlo the two "expected values" with low-dimensional calculations

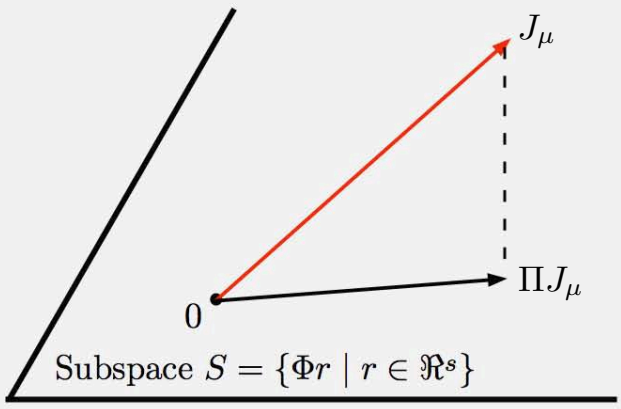
$$\hat{r}_k = \left(\sum_{t=1}^k \phi(i_t)\phi(i_t)'\right)^{-1} \sum_{t=1}^k \phi(i_t) J_\mu(i_t)$$

- Equivalent least squares alternative calculation:

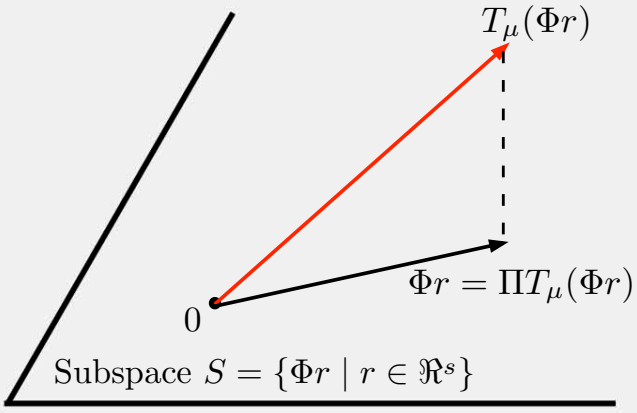
$$\hat{r}_k = \arg\min_{r\in\Re^s} \sum_{t=1}^k \big(\phi(i_t)'r - J_\mu(i_t)\big)^2$$

INDIRECT POLICY EVALUATION

- An example: Solve the projected equation $\Phi r = \Pi T_\mu(\Phi r)$ where Π is projection w/ respect to a suitable weighted Euclidean norm (Galerkin approx.



Direct Method: Projection of cost vector J_μ



Indirect Method: Solving a projected form of Bellman's equation

- Solution methods that use simulation (to manage the calculation of Π)
  - TD(λ): Stochastic iterative algorithm for solving $\Phi r = \Pi T_\mu(\Phi r)$
  - LSTD(λ): Solves a simulation-based approximation w/ a standard solver
  - LSPE(λ): A simulation-based form of projected value iteration; essentially

$$\Phi r_{k+1} = \Pi T_\mu(\Phi r_k) + \text{ simulation noise}$$

BELLMAN EQUATION ERROR METHODS

- Another example of indirect approximate policy evaluation:

$$\min_r \|\Phi r - T_\mu(\Phi r)\|_\xi^2 \qquad (*)$$

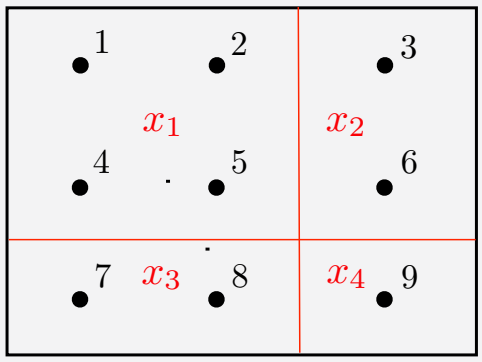
where $\|\cdot\|_\xi$ is Euclidean norm, weighted with respect to some distribution ξ

- It is closely related to the projected equation approach (with a special choice of projection norm)

- Several ways to implement projected equation and Bellman error methods by simulation. They involve:
  - Generating many random samples of states i_k using the distribution ξ
  - Generating many samples of transitions (i_k, j_k) using the policy μ
  - Form a simulation-based approximation of the optimality condition for projection problem or problem (*) (use sample averages in place of inner products)
  - Solve the Monte-Carlo approximation of the optimality condition
- Issues for indirect methods: How to generate the samples? How to calculate r^* efficiently?

ANOTHER INDIRECT METHOD: AGGREGATION

- An example: Group similar states together into "aggregate states" $x_1, \ldots, x_s$; assign a common cost r_i to each group x_i. A linear architecture called hard aggregation.



$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Solve an "aggregate" DP problem to obtain $r = (r_1, \ldots, r_s)$.
- More general/mathematical view: Solve

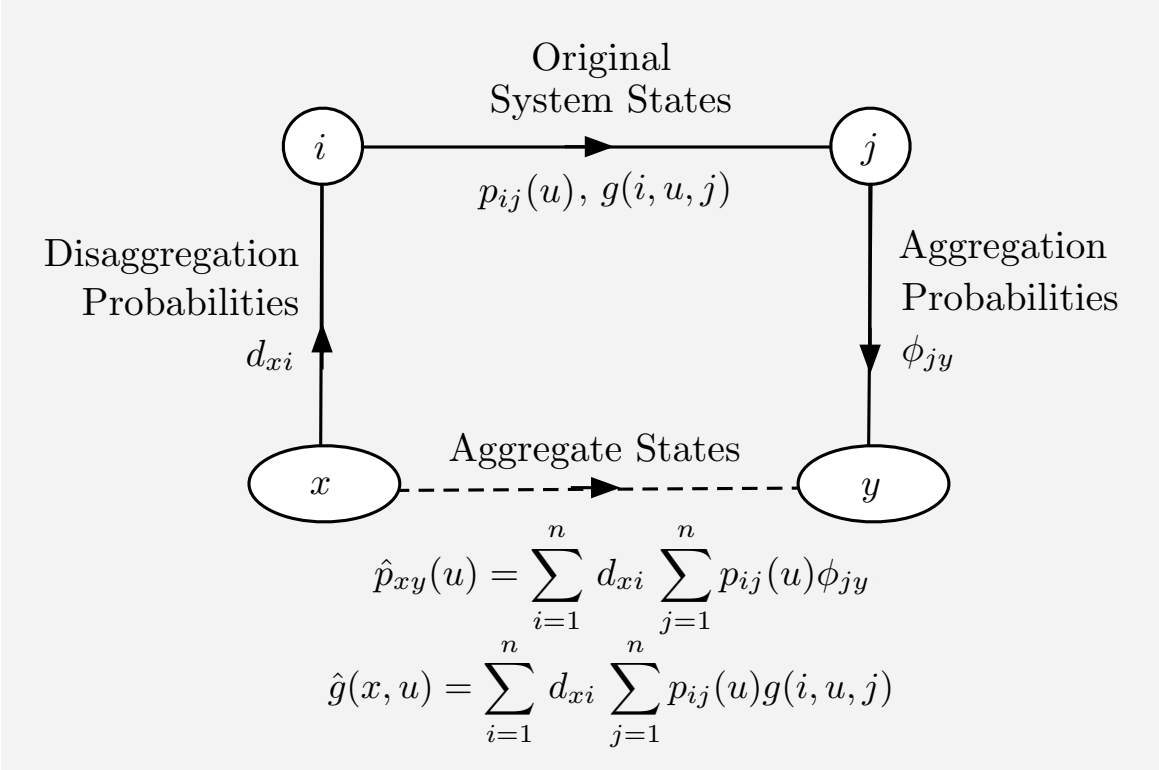
$$\Phi r = \Phi D T_\mu(\Phi r)$$

where the rows of D and Φ are prob. distributions (e.g., D and Φ "aggregate" rows and columns of the linear system $J = T_\mu J$)

- Compare with projected equation $\Phi r = \Pi T_\mu(\Phi r)$.

Note: ΦD is a projection in some interesting cases

AGGREGATION AS PROBLEM APPROXIMATION



- Aggregation can be viewed as a systematic approach for problem approx. Main elements:
  - Solve (exactly or approximately) the "aggregate" problem by any kind of VI or PI method (including simulation-based methods)
  - Use the optimal cost of the aggregate problem to approximate the optimal cost of the original problem

- Because an exact PI algorithm is used to solve the approximate/aggregate problem the method behaves more regularly than the projected equation approach

THEORETICAL BASIS OF APPROXIMATE PI

- If policies are approximately evaluated using an approximation architecture such that

$$\max_i |\tilde{J}(i, r_k) - J_{\mu^k}(i)| \leq \delta, \qquad k = 0, 1, \ldots$$

- If policy improvement is also approximate,

$$\max_i |(T_{\mu^{k+1}} \tilde{J})(i, r_k) - (T\tilde{J})(i, r_k)| \leq \epsilon, \qquad k = 0, 1, \ldots$$

- Error bound: The sequence $\{\mu^k\}$ generated by approximate policy iteration satisfies

$$\limsup_{k \to \infty} \max_i \left(J_{\mu^k}(i) - J^*(i)\right) \leq \frac{\epsilon + 2\alpha\delta}{(1-\alpha)^2}$$

- Typical practical behavior: The method makes steady progress up to a point and then the iterates J_{μ^k} oscillate within a neighborhood of J^*.
- Oscillations are quite unpredictable.
  - Bad examples of oscillations are known.
  - In practice oscillations between policies is probably not the major concern.
  - In aggregation case, there are no oscillations

THE ISSUE OF EXPLORATION

- To evaluate a policy μ, we need to generate cost samples using that policy - this biases the simulation by underrepresenting states that are unlikely to occur under μ
- Cost-to-go estimates of underrepresented states may be highly inaccurate
- This seriously impacts the improved policy $\overline{\mu}$
- This is known as inadequate exploration - a particularly acute difficulty when the randomness embodied in the transition probabilities is "relatively small" (e.g., a deterministic system)
- Some remedies:
  - Frequently restart the simulation and ensure that the initial states employed form a rich and representative subset
  - Occasionally generate transitions that use a randomly selected control rather than the one dictated by the policy μ
  - Other methods: Use two Markov chains (one is the chain of the policy and is used to generate the transition sequence, the other is used to generate the state sequence).

APPROXIMATING Q-FACTORS

- Given $\tilde{J}(i;r)$, policy improvement requires a model [knowledge of $p_{ij}(u)$ for all $u \in U(i)$]
- Model-free alternative: Approximate Q-factors

$$\tilde{Q}(i,u;r) \approx \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J_\mu(j)\big)$$

and use for policy improvement the minimization

$$\overline{\mu}(i) \in \arg\min_{u \in U(i)} \tilde{Q}(i,u;r)$$

- r is an adjustable parameter vector and $\tilde{Q}(i,u;r)$ is a parametric architecture, such as

$$\tilde{Q}(i,u;r) = \sum_{m=1}^{s} r_m \phi_m(i,u)$$

- We can adapt any of the cost approximation approaches, e.g., projected equations, aggregation
- Use the Markov chain with states (i,u), so $p_{ij}(\mu(i))$ is the transition prob. to $(j,\mu(i))$, 0 to other (j,u')
- Major concern: Acutely diminished exploration

STOCHASTIC ALGORITHMS: GENERALITIES

- Consider solution of a linear equation $x = b + Ax$ by using m simulation samples $b + w_k$ and $A + W_k$, $k = 1, \ldots, m$, where w_k, W_k are random, e.g., "simulation noise"

- Think of $x = b + Ax$ as approximate policy evaluation (projected or aggregation equations)

- Stoch. approx. (SA) approach: For $k = 1, \ldots, m$

$$x_{k+1} = (1 - \gamma_k)x_k + \gamma_k\big((b + w_k) + (A + W_k)x_k\big)$$

- Monte Carlo estimation (MCE) approach: Form Monte Carlo estimates of b and A

$$b_m = \frac{1}{m}\sum_{k=1}^{m}(b + w_k), \qquad A_m = \frac{1}{m}\sum_{k=1}^{m}(A + W_k)$$

Then solve $x = b_m + A_m x$ by matrix inversion

$$x_m = (1 - A_m)^{-1} b_m$$

or iteratively

- TD(λ) and Q-learning are SA methods
- LSTD(λ) and LSPE(λ) are MCE methods

6.231 Dynamic Programming and Stochastic Control
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