6.094
Introduction to programming in MATLAB

Lecture 3 : Solving Equations and Curve Fitting

Danilo Šćepanović

IAP 2008

Homework 2 Recap

- How long did it take?

- Using min with matrices:
  ```
  » a=[3 7 5;1 9 10; 30 -1 2];
  » b=min(a); % returns the min of each column
  » m=min(b); % returns min of entire a matrix
  » m=min(min(a)); % same as above
  » m=min(a(:)); % makes a a vector, then gets min
  ```
- Common mistake:
  ```
  » [m,n]=find(min(a)); % think about what happens
  ```
- How to make and run a function: save the file, then call it from the command window like any other function. No need to 'compile' or make it official in any other way

Outline

(1) Linear Algebra

(2) Polynomials

(3) Optimization

(4) Differentiation/Integration

(5) Differential Equations

Systems of Linear Equations

MATLAB makes linear algebra fun!

- Given a system of linear equations
  - x+2y-3z=5
  - -3x-y+z=-8
  - x-y+z=0
- Construct matrices so the system is described by Ax=b

```
» A=[1 2 -3;-3 -1 1;1 -1 1];
» b=[5;-8;0];
```

- And solve with a single line of code!

```
» x=A\b;
```

  - x is a 3x1 vector containing the values of x, y, and z

- The \ will work with square or rectangular systems.
- Gives least squares solution for rectangular systems. Solution depends on whether the system is over or underdetermined.

More Linear Algebra

- Given a matrix
  » `mat=[1 2 -3;-3 -1 1;1 -1 1];`

- Calculate the rank of a matrix
  » `r=rank(mat);`
    - the number of linearly independent rows or columns

- Calculate the determinant
  » `d=det(mat);`
    - mat must be square
    - if determinant is nonzero, matrix is invertible

- Get the matrix inverse
  » `E=inv(mat);`
    - if an equation is of the form A*x=b with A a square matrix, x=A\b is the same as x=inv(A)*b

Matrix Decompositions

- MATLAB has built-in matrix decomposition methods

- The most common ones are
  - » `[V,D]=eig(X)`
    - ➢ Eigenvalue decomposition
  - » `[U,S,V]=svd(X)`
    - ➢ Singular value decomposition
  - » `[Q,R]=qr(X)`
    - ➢ QR decomposition

Exercise: Linear Algebra

- Solve the following systems of equations:

  - System 1:

$$
\begin{aligned}
x + 4y &= 34 \\
-3x + y &= 2
\end{aligned}
$$

  - System 2:

$$
\begin{aligned}
2x - 2y &= 4 \\
-x + y &= 3 \\
3x + 4y &= 2
\end{aligned}
$$

Exercise: Linear Algebra

- Solve the following systems of equations:
  - System 1:

$$x + 4y = 34$$
$$-3x + y = 2$$

```
» A=[1 4;-3 1];
» b=[34;2];
» rank(A)
» x=inv(A)*b;
```

  - System 2:

$$2x - 2y = 4$$
$$-x + y = 3$$
$$3x + 4y = 2$$

```
» A=[2 -2;-1 1;3 4];
» b=[4;3;2];
» rank(A)
```

  - rectangular matrix

```
» x1=A\b;
```

  - gives least squares solution

```
» error=abs(A*x1-b)
```

Outline

(1) Linear Algebra

(2) Polynomials

(3) Optimization

(4) Differentiation/Integration

(5) Differential Equations

Polynomials

- Many functions can be well described by a high-order polynomial

- MATLAB represents a polynomials by a vector of coefficients
  - if vector P describes a polynomial

$$ax^3+bx^2+cx+d$$

P(1) P(2) P(3) P(4)

- P=[1 0 -2] represents the polynomial x^2-2

- P=[2 0 0 0] represents the polynomial $2x^3$

Polynomial Operations

- P is a vector of length N+1 describing an N-th order polynomial
- To get the roots of a polynomial
  - » `r=roots(P)`
    - ➢ r is a vector of length N

- Can also get the polynomial from the roots
  - » `P=poly(r)`
    - ➢ r is a vector length N

- To evaluate a polynomial at a point
  - » `y0=polyval(P,x0)`
    - ➢ x0 is a single value; y0 is a single value

- To evaluate a polynomial at many points
  - » `y=polyval(P,x)`
    - ➢ x is a vector; y is a vector of the same size

Polynomial Fitting

- MATLAB makes it very easy to fit polynomials to data
- Given data vectors X=[-1 0 2] and Y=[0 -1 3]
  - » **p2=polyfit(X,Y,2);**
    - ➢ finds the best second order polynomial that fits the points (-1,0),(0,-1), and (2,3)
    - ➢ see **help polyfit** for more information
  - » **plot(X,Y,'o', 'MarkerSize', 10);**
  - » **hold on;**
  - » **x = -3:.01:3;**
  - » **plot(x,polyval(p2,x), 'r--');**

Exercise: Polynomial Fitting

- Evaluate $y = x^2$ for x=-4:0.1:4.

- Add random noise to these samples. Use **randn**. Plot the noisy signal with **.** markers

- Fit a 2^{nd} degree polynomial to the noisy data

- Plot the fitted polynomial on the same plot, using the same x values and a red line

Exercise: Polynomial Fitting

- Evaluate $y = x^2$ for x=-4:0.1:4.
  ```
  » x=-4:0.1:4;
  » y=x.^2;
  ```

- Add random noise to these samples. Use **randn**. Plot the noisy signal with **.** markers
  ```
  » y=y+randn(size(y));
  » plot(x,y,'.');
  ```
- Fit a 2nd degree polynomial to the noisy data
  ```
  » p=polyfit(x,y,2);
  ```
- Plot the fitted polynomial on the same plot, using the same x values and a red line
  ```
  » hold on;
  » plot(x,polyval(p,x),'r')
  ```

Outline

(1) Linear Algebra

(2) Polynomials

(3) Optimization

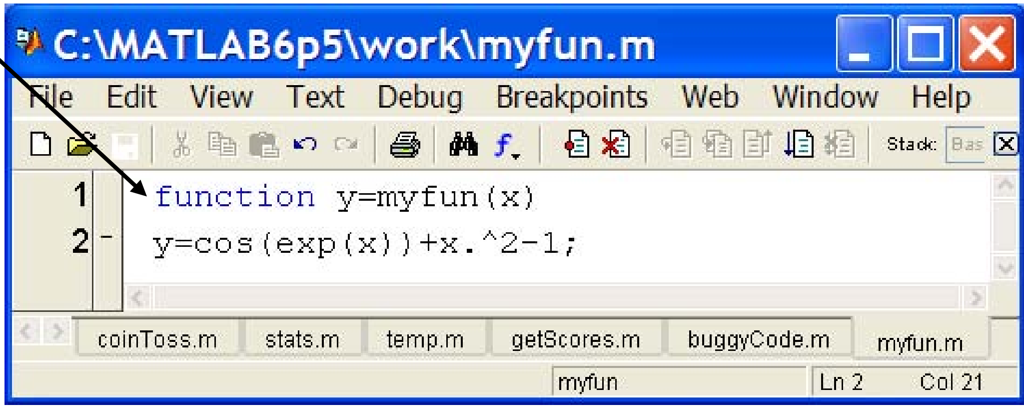
(4) Differentiation/Integration

(5) Differential Equations

Nonlinear Root Finding

- Many real-world problems require us to solve f(x)=0
- Can use **fzero** to calculate roots for *any* arbitrary function

- **fzero** needs a function passed to it.
- We will see this more and more as we delve into solving equations.
- Make a separate function file
  - » **x=fzero('myfun',1)**
  - » **x=fzero(@myfun,1)**
    - 1 specifies a point close to where you think the root is



Courtesy of The MathWorks, Inc. Used with permission.

Minimizing a Function

- **fminbnd**: minimizing a function over a bounded interval
  - » `x=fminbnd('myfun',-1,2);`
    - ➢ myfun takes a scalar input and returns a scalar output
    - ➢ myfun(x) will be the minimum of myfun for $-1 \leq x \leq 2$
- **fminsearch**: unconstrained interval
  - » `x=fminsearch('myfun',.5)`
    - ➢ finds the local minimum of myfun starting at x=0.5

Anonymous Functions

- You do not have to make a separate function file
  - » `x=fzero(@myfun,1)`
    - What if myfun is really simple?

- Instead, you can make an anonymous function
  - » `x=fzero(@(x)(cos(exp(x))+x^2-1),  1  );`

    input   function to evaluate

  - » `x=fminbnd(@(x) (cos(exp(x))+x^2-1),-1,2);`

Optimization Toolbox

- If you are familiar with optimization methods, use the optimization toolbox

- Useful for larger, more structured optimization problems

- Sample functions (see **`help`** for more info)
  - » **`linprog`**
    - ➢ linear programming using interior point methods
  - » **`quadprog`**
    - ➢ quadratic programming solver
  - » **`fmincon`**
    - ➢ constrained nonlinear optimization

Exercise: Min-Finding

- Find the minimum of the function $f(x) = \cos(4x)\sin(10x)e^{-|x|}$ over the range −π to π. Use **fminbnd**.
- Plot the function on this range to check that this is the minimum.

Exercise: Min-Finding

- Find the minimum of the function $f(x)=\cos(4x)\sin(10x)e^{-|x|}$ over the range −π to π. Use **`fminbnd`**.
- Plot the function on this range to check that this is the minimum.

- Make the following function:
  - » `function y=myFun(x)`
  - » `y=cos(4*x).*sin(10*x).*exp(-abs(x));`

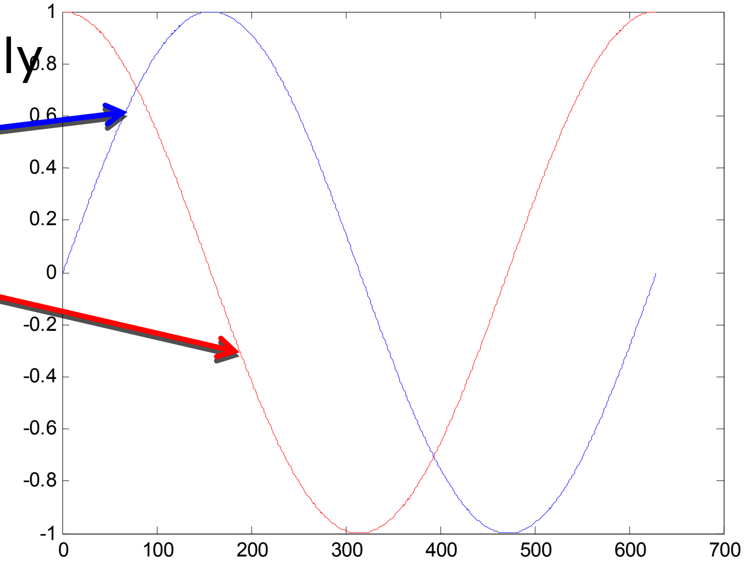
- Find the minimum in the command window:
  - » `x0=fminbnd('myFun',-pi,pi);`
- Plot to check if it's right
  - » `figure; x=-pi:.01:pi; plot(x,myFun(x));`

Outline

(1) Linear Algebra
(2) Polynomials
(3) Optimization
(4) Differentiation/Integration
(5) Differential Equations

Numerical Differentiation

- MATLAB can 'differentiate' numerically
  - » `x=0:0.01:2*pi;`
  - » `y=sin(x);`
  - » `dydx=diff(y)./diff(x);`
    - ➢ diff computes the first difference



- Can also operate on matrices
  - » `mat=[1 3 5;4 8 6];`
  - » `dm=diff(mat,1,2)`
    - ➢ first difference of mat along the 2nd dimension, dm=[2 2;4 -2]
    - ➢ see **help** for more details
    - ➢ The opposite of `diff` is the cumulative sum `cumsum`

- 2D gradient
  - » `[dx,dy]=gradient(mat);`

Numerical Integration

- MATLAB contains common integration methods
- Adaptive Simpson's quadrature (input is a function)
  - » `q=quad('myFun',0,10);`
    - q is the integral of the function `myFun` from 0 to 10
  - » `q2=quad(@(x) sin(x)*x,0,pi)`
    - q2 is the integral of `sin(x)*x` from 0 to pi
- Trapezoidal rule (input is a vector)
  - » `x=0:0.01:pi;`
  - » `z=trapz(x,sin(x));`
    - z is the integral of sin(x) from 0 to pi
  - » `z2=trapz(x,sqrt(exp(x))./x)`
    - z2 is the integral of $\sqrt{e^x}/x$ from 0 to pi

Outline

(1) Linear Algebra

(2) Polynomials

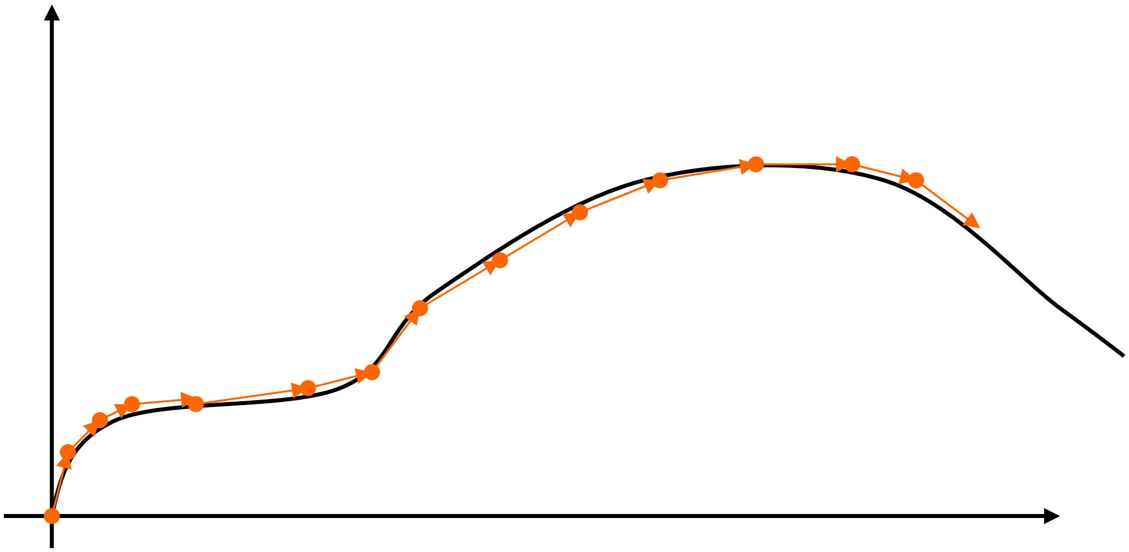
(3) Optimization

(4) Differentiation/Integration

(5) Differential Equations

ODE Solvers: Method

- Given a differential equation, the solution can be found by integration:



- ➢ Evaluate the derivative at a point and approximate by straight line
- ➢ Errors accumulate!
- ➢ Variable timestep can decrease the number of iterations

ODE Solvers: MATLAB

- MATLAB contains implementations of common ODE solvers

- Using the correct ODE solver can save you lots of time and give more accurate results
  - » **`ode23`**
    - ➢ Low-order solver. Use when integrating over small intervals or when accuracy is less important than speed
  - » **`ode45`**
    - ➢ High order (Runge-Kutta) solver. High accuracy and reasonable speed. Most commonly used.
  - » **`ode15s`**
    - ➢ Stiff ODE solver (Gear's algorithm), use when the diff eq's have time constants that vary by orders of magnitude

ODE Solvers: Standard Syntax

- To use standard options and variable time step

```
» [t,y]=ode45('myODE',[0,10],[1;0])
```

ODE integrator: 23, 45, 15s

ODE function

Time range

Initial conditions

- Inputs:
  - ➢ ODE function name (or anonymous function). This function takes inputs (t,y), and returns dy/dt
  - ➢ Time interval: 2-element vector specifying initial and final time
  - ➢ Initial conditions: column vector with an initial condition for each ODE. This is the first input to the ODE function
- Outputs:
  - ➢ t contains the time points
  - ➢ y contains the corresponding values of the integrated variables.

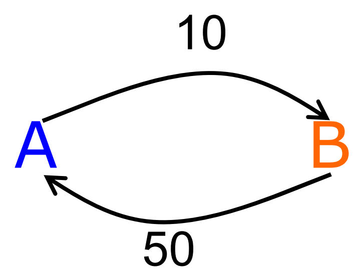
ODE Function

- The ODE function must return the value of the derivative at a given time and function value

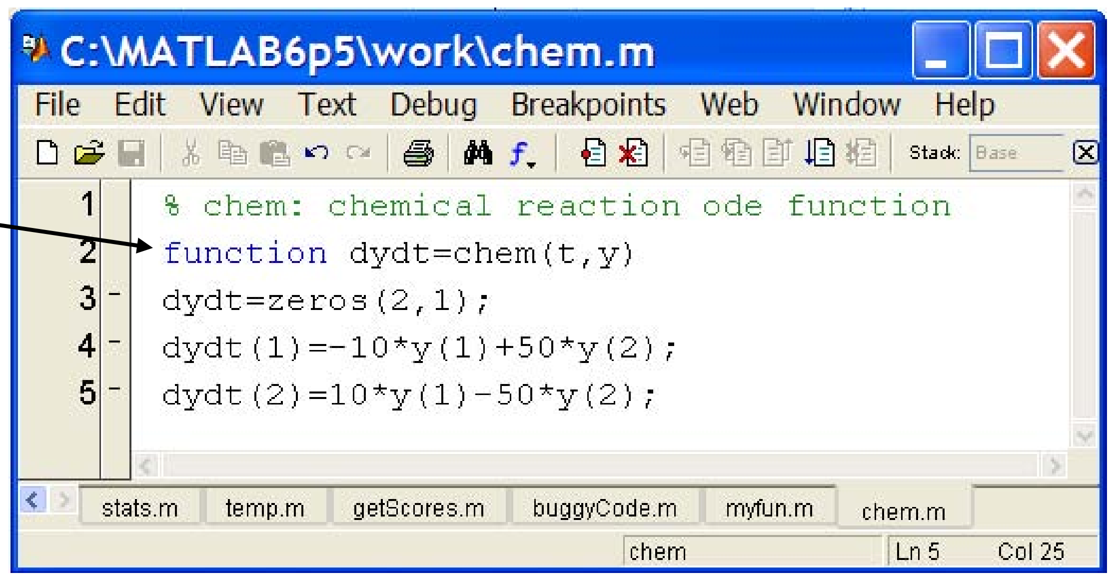
- Example: chemical reaction
  - Two equations

$$\frac{dA}{dt} = -10A + 50B$$

$$\frac{dB}{dt} = 10A - 50B$$



  - ODE file:
    - y has [A;B]
    - dydt has [dA/dt;dB/dt]



ODE Function: viewing results

- To solve and plot the ODEs on the previous slide:

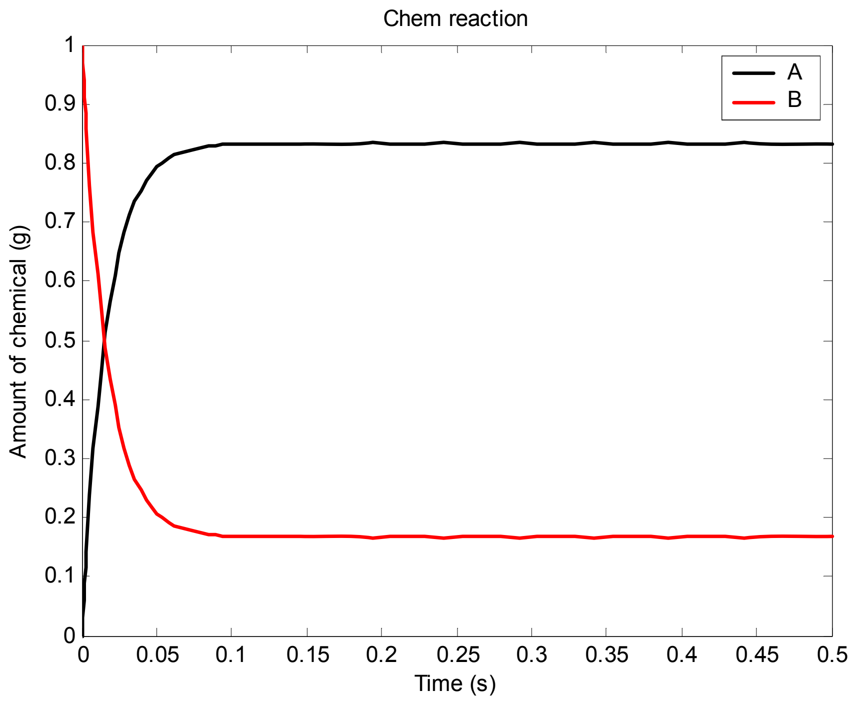
```
» [t,y]=ode45('chem',[0 0.5],[0 1]);
```

  - ➢ assumes that only chemical B exists initially

```
» plot(t,y(:,1),'k','LineWidth',1.5);
» hold on;
» plot(t,y(:,2),'r','LineWidth',1.5);
» legend('A','B');
» xlabel('Time (s)');
» ylabel('Amount of chemical (g)');
» title('Chem reaction');
```

ODE Function: viewing results

- The code on the previous slide produces this figure



Higher Order Equations

- Must make into a system of first-order equations to use ODE solvers
- Nonlinear is OK!
- Pendulum example:

$$\ddot{\theta} + \frac{g}{L} sin(\theta) = 0$$

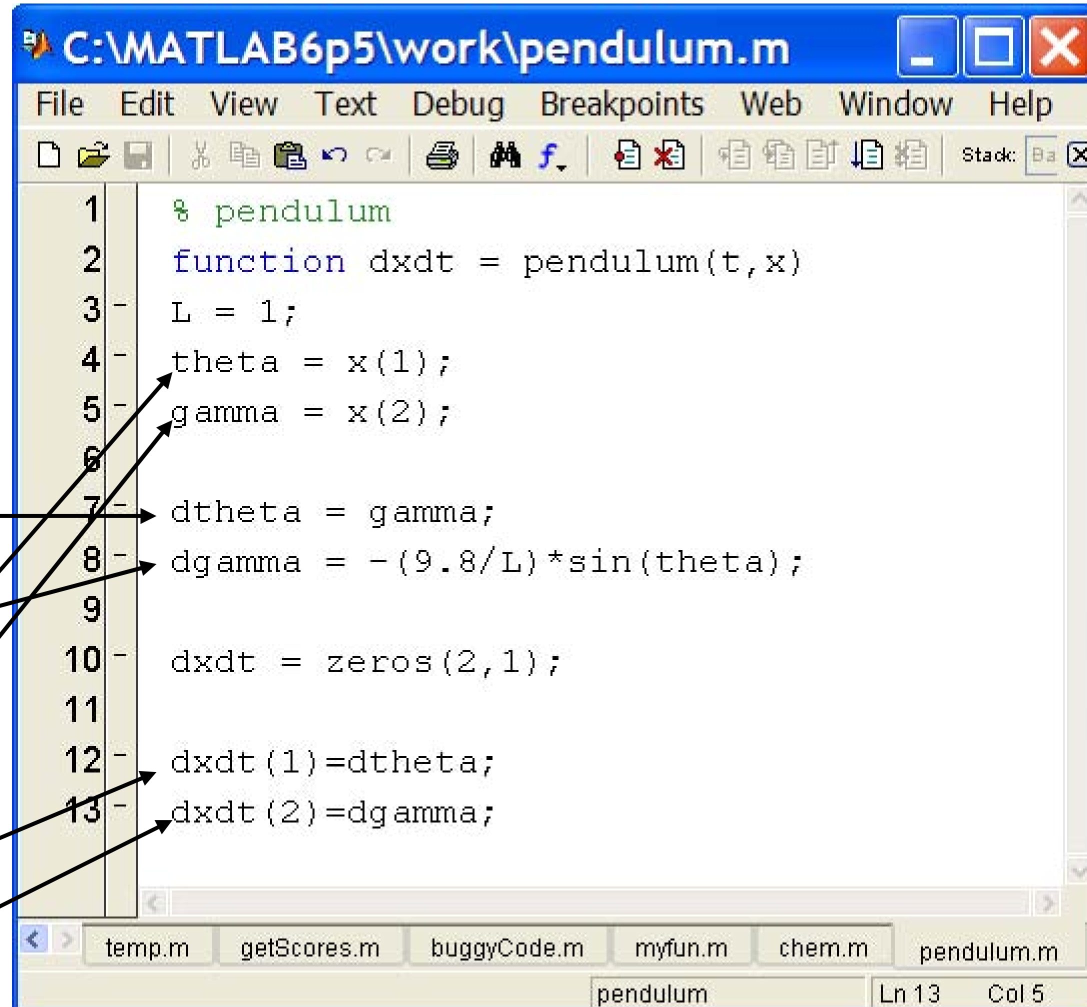
$$\ddot{\theta} = -\frac{g}{L} sin(\theta)$$

$$let \quad \dot{\theta} = \gamma$$

$$\dot{\gamma} = -\frac{g}{L} sin(\theta)$$

$$\vec{x} = \begin{bmatrix} \theta \\ \gamma \end{bmatrix}$$

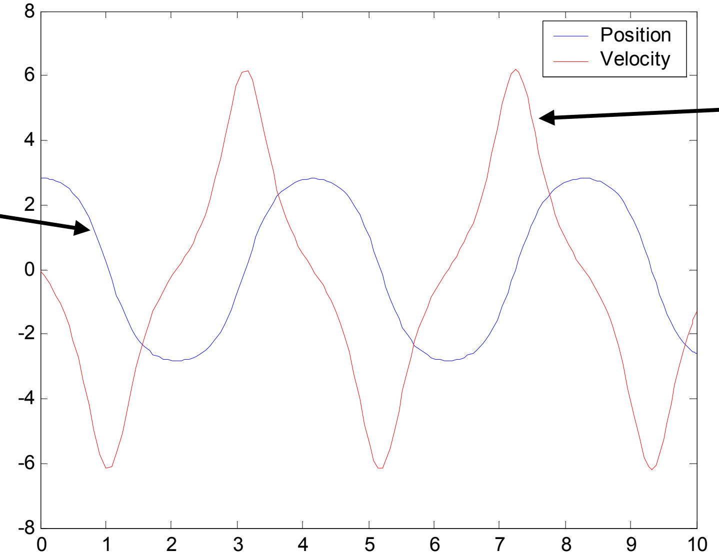
$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \end{bmatrix}$$



Plotting the Output

- We can solve for the position and velocity of the pendulum:
  - » `[t,x]=ode45('pendulum',[0 10],[0.9*pi 0]);`
    - ➢ assume pendulum is almost horizontal
  - » `plot(t,x(:,1));`
  - » `hold on;`
  - » `plot(t,x(:,2),'r');`
  - » `legend('Position','Velocity');`

Position in terms of angle (rad)



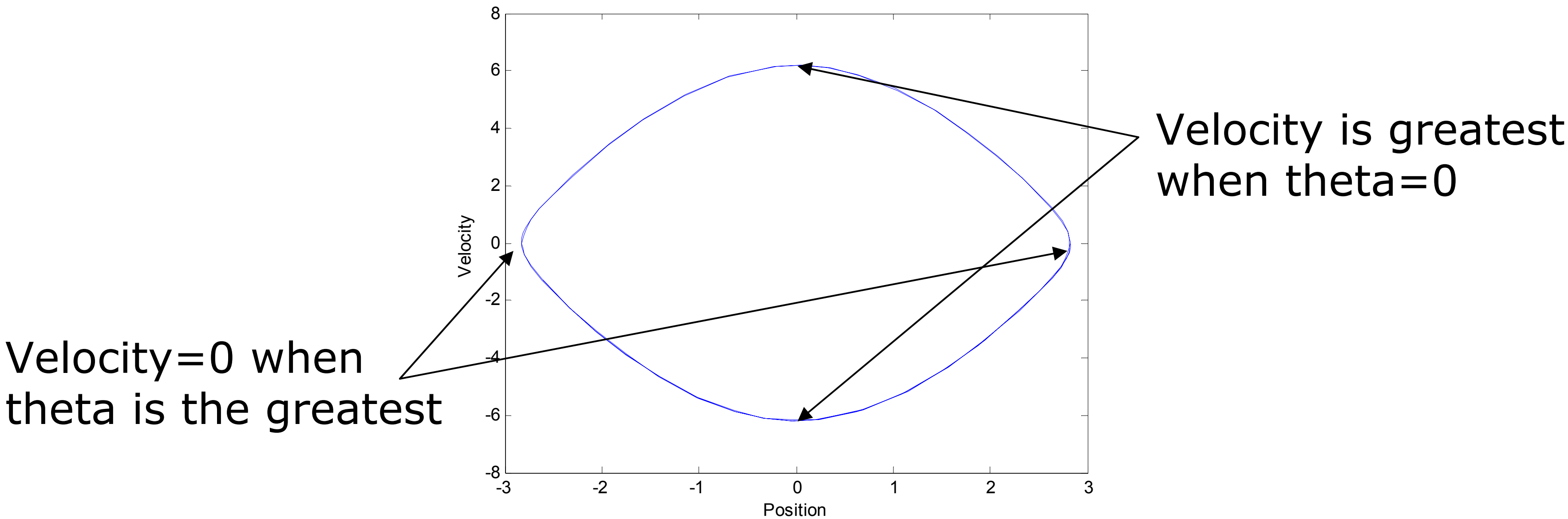
Velocity (m/s)

Plotting the Output

- Or we can plot in the phase plane:

```
» plot(x(:,1),x(:,2));
» xlabel('Position');
» yLabel('Velocity');
```

- The phase plane is just a plot of one variable versus the other:



ODE Solvers: Custom Options

- MATLAB's ODE solvers use a variable timestep
- Sometimes a fixed timestep is desirable
  » `[t,y]=ode45('chem',[0:0.001:0.5],[0 1]);`
    - Specify the timestep by giving a vector of times
    - The function value will be returned at the specified points
    - Fixed timestep is usually slower because function values are interpolated to give values at the desired timepoints

- You can customize the error tolerances using odeset
  » `options=odeset('RelTol',1e-6,'AbsTol',1e-10);`
  » `[t,y]=ode45('chem',[0 0.5],[0 1],options);`
    - This guarantees that the error at each step is less than **`RelTol`** times the value at that step, and less than **`AbsTol`**
    - Decreasing error tolerance can considerably slow the solver
    - See doc odeset for a list of options you can customize

Exercise: ODE

- Use `ode45` to solve for $y(t)$ on the range t=[0 10], with initial condition $y(0)=10$ and $dy/dt=-t\,y/10$
- Plot the result.

Exercise: ODE

- Use **ode45** to solve for $y(t)$ on the range t=[0 10], with initial condition $y(0)=10$ and $dy/dt=-t\,y/10$
- Plot the result.

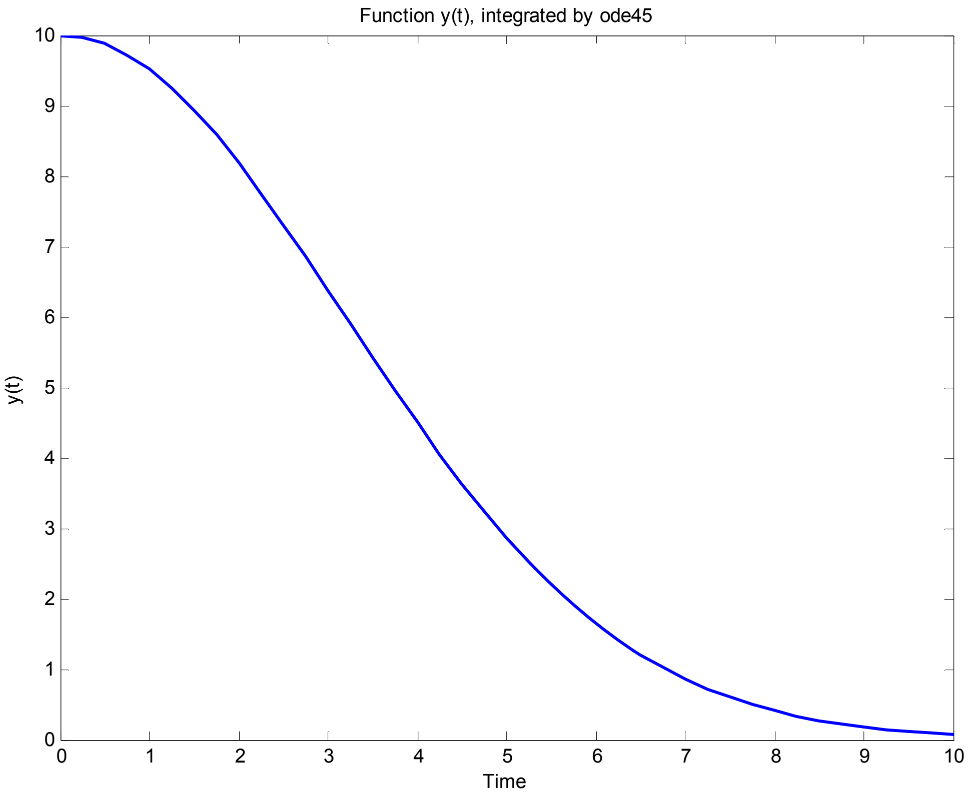
- Make the following function
  ```
  » function dydt=odefun(t,y)
  » dydt=-t*y/10;
  ```
- Integrate the ODE function and plot the result
  ```
  » [t,y]=ode45('odefun',[0 10],10);
  ```

- Alternatively, use an anonymous function
  ```
  » [t,y]=ode45(@(t,y) -t*y/10,[0 10],10);
  ```

- Plot the result
  ```
  » plot(t,y);xlabel('Time');ylabel('y(t)');
  ```

Exercise: ODE

- The integrated function looks like this:



End of Lecture 3

(1) Linear Algebra

(2) Polynomials

(3) Optimization

(4) Differentiation/Integration

(5) Differential Equations

We're almost done!

6.094 Introduction to MATLAB®
January (IAP) 2010