Homework 5 additional problems

1. *Heuristic suboptimal solution for Boolean LP.* This exercise builds on exercises 4.15 and 5.13 in *Convex Optimization*, which involve the Boolean LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1,\ldots,n, \end{array}$$

with optimal value $p^\star$. Let $x^{\rm rlx}$ be a solution of the LP relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \preceq x \preceq \mathbf{1}, \end{array}$$

so $L = c^T x^{\rm rlx}$ is a lower bound on $p^\star$. The relaxed solution $x^{\rm rlx}$ can also be used to guess a Boolean point $\hat{x}$, by rounding its entries, based on a threshold $t \in [0,1]$:

$$\hat{x}_i = \begin{cases} 1 & x_i^{\rm rlx} \geq t \\ 0 & \text{otherwise}, \end{cases}$$

for $i = 1,\ldots,n$. Evidently $\hat{x}$ is Boolean (*i.e.*, has entries in $\{0,1\}$). If it is feasible for the Boolean LP, *i.e.*, if $A\hat{x} \preceq b$, then it can be considered a guess at a good, if not optimal, point for the Boolean LP. Its objective value, $U = c^T\hat{x}$, is an upper bound on $p^\star$. If U and L are close, then $\hat{x}$ is nearly optimal; specifically, $\hat{x}$ cannot be more than $(U - L)$-suboptimal for the Boolean LP.

This rounding need not work; indeed, it can happen that for all threshold values, $\hat{x}$ is infeasible. But for some problem instances, it can work well.

Of course, there are many variations on this simple scheme for (possibly) constructing a feasible, good point from $x^{\rm rlx}$.

Finally, we get to the problem. Generate problem data using

```
rand('state',0);
n=100;
m=300;
A=rand(m,n);
b=A*ones(n,1)/2;
c=-rand(n,1);
```

You can think of x_i as a job we either accept or decline, and $-c_i$ as the (positive) revenue we generate if we accept job i. We can think of $Ax \preceq b$ as a set of limits on

m resources. A_{ij}, which is positive, is the amount of resource i consumed if we accept job j; b_i, which is positive, is the amount of resource i available.

Find a solution of the relaxed LP and examine its entries. Note the associated lower bound L. Carry out threshold rounding for (say) 100 values of t, uniformly spaced over $[0,1]$. For each value of t, note the objective value $c^T\hat{x}$ and the maximum constraint violation $\max_i(A\hat{x}-b)_i$. Plot the objective value and the maximum violation versus t. Be sure to indicate on the plot the values of t for which $\hat{x}$ is feasible, and those for which it is not.

Find a value of t for which $\hat{x}$ is feasible, and gives minimum objective value, and note the associated upper bound U. Give the gap $U-L$ between the upper bound on $p^\star$ and the lower bound on $p^\star$. If you define vectors `obj` and `maxviol`, you can find the upper bound as `U=min(obj(find(maxviol<=0)))`.

2. *Three measures of the spread of a group of numbers.* For $x \in \mathbf{R}^n$, we define three functions that measure the spread or width of the set of its elements (or coefficients). The first function is the *spread*, defined as

$$\phi_{\mathrm{sprd}}(x) = \max_{i=1,\ldots,n} x_i - \min_{i=1,\ldots,n} x_i.$$

This is the width of the smallest interval that contains all the elements of x.

The second function is the *standard deviation*, defined as

$$\phi_{\mathrm{stdev}}(x) = \left(\frac{1}{n}\sum_{i=1}^n x_i^2 - \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2\right)^{1/2}.$$

This is the statistical standard deviation of a random variable that takes the values $x_1,\ldots,x_n$, each with probability $1/n$.

The third function is the average absolute deviation from the median of the values:

$$\phi_{\mathrm{aamd}}(x) = (1/n)\sum_{i=1}^n |x_i - \mathrm{med}(x)|,$$

where $\mathrm{med}(x)$ denotes the median of the components of x, defined as follows. If $n = 2k-1$ is odd, then the median is defined as the value of middle entry when the components are sorted, *i.e.*, $\mathrm{med}(x) = x_{[k]}$, the kth largest element among the values $x_1,\ldots,x_n$. If $n=2k$ is even, we define the median as the average of the two middle values, *i.e.*, $\mathrm{med}(x) = (x_{[k]} + x_{[k+1]})/2$.

Each of these functions measures the spread of the values of the entries of x; for example, each function is zero if and only if all components of x are equal, and each function is unaffected if a constant is added to each component of x.

Which of these three functions is convex? For each one, either show that it is convex, or give a counterexample showing it is not convex. By a counterexample, we mean a specific x and y such that Jensen's inequality fails, *i.e.*, $\phi((x+y)/2) > (\phi(x)+\phi(y))/2$.

3. *Minimax rational fit to the exponential.* (See exercise 6.9 of *Convex Optimization.*) We consider the specific problem instance with data

$$t_i = -3 + 6(i-1)/(k-1), \quad y_i = e^{t_i}, \quad i = 1, \ldots, k,$$

where $k = 201$. (In other words, the data are obtained by uniformly sampling the exponential function over the interval $[-3, 3]$.) Find a function of the form

$$f(t) = \frac{a_0 + a_1 t + a_2 t^2}{1 + b_1 t + b_2 t^2}$$

that minimizes $\max_{i=1,\ldots,k} |f(t_i) - y_i|$. (We require that $1 + b_1 t_i + b_2 t_i^2 > 0$ for $i = 1, \ldots, k$.)

Find optimal values of $a_0,\ a_1,\ a_2,\ b_1,\ b_2$, and give the optimal objective value, computed to an accuracy of 0.001. Plot the data and the optimal rational function fit on the same plot. On a different plot, give the fitting error, *i.e.*, $f(t_i) - y_i$.

Hint. You can use `strcmp(cvx_status,'Solved')`, after `cvx_end`, to check if a feasibility problem is feasible.

4. *Complex least-norm problem.* We consider the complex least ℓ_p-norm problem

$$\begin{array}{ll} \text{minimize} & \|x\|_p \\ \text{subject to} & Ax = b, \end{array}$$

where $A \in \mathbf{C}^{m \times n}$, $b \in \mathbf{C}^m$, and the variable is $x \in \mathbf{C}^n$. Here $\|\cdot\|_p$ denotes the ℓ_p-norm on $\mathbf{C}^n$, defined as

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

for $p \geq 1$, and $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We assume A is full rank, and $m < n$.

(a) Formulate the complex least ℓ_2-norm problem as a least ℓ_2-norm problem with real problem data and variable. *Hint.* Use $z = (\Re x, \Im x) \in \mathbf{R}^{2n}$ as the variable.

(b) Formulate the complex least ℓ_∞-norm problem as an SOCP.

(c) Solve a random instance of both problems with $m = 30$ and $n = 100$. To generate the matrix A, you can use the Matlab command `A = randn(m,n) + i*randn(m,n)`. Similarly, use `b = randn(m,1) + i*randn(m,1)` to generate the vector b. Use the Matlab command `scatter` to plot the optimal solutions of the two problems on the complex plane, and comment (briefly) on what you observe. You can solve the problems using the CVX functions `norm(x,2)` and `norm(x,inf)`, which are overloaded to handle complex arguments. To utilize this feature, you will need to declare variables to be `complex` in the `variable` statement. (In particular, you do not have to manually form or solve the SOCP from part (b).)

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