
Problem Set 2 Solutions

This problem set is due at **9:00pm** on **Wednesday, February 22, 2012**.

Problem 2-1. Unique Minimum Spanning Trees

In Lecture 3, we saw the claim that any weighted undirected graph with distinct edge weights has exactly one minimum spanning tree. In this problem, your goal will be to show that this claim is true by proving a more general theorem.

Let $G = (V, E)$ be a connected, weighted undirected graph whose edges weight may or may not be distinct. Given a cut $(S, V - S)$ of S , recall that an edge $(u, v) \in E$ is said to cross the cut if exactly one of u and v is in S . An edge (u, v) is a light edge crossing the cut $(S, V - S)$ if its weight is the smallest of any edge crossing $(S, V - S)$. There can be more than one light edge crossing a single cut in the case of ties. An edge (u, v) is a unique light edge if there are no ties.

- (a) Show that if an edge (u, v) is the unique light edge crossing some cut of the connected, weighted undirected graph G , then (u, v) must be included in *all* minimum spanning trees of G .

Solution: Suppose for contradiction that there is some MST T that does not contain the light edge (u, v) crossing a cut C . Because T is a connected tree, there is some path in T connecting the vertices u and v , and therefore adding the edge (u, v) to T forms a cycle.

Following this cycle, there must be at least one other edge in the cycle crossing the cut C . Removing this edge, and leaving (u, v) , we now have a tree with a lower total weight than the original T . Because T was assumed to be an MST, this is a contradiction. Therefore, any unique light edge of a cut must be part of every MST.

- (b) Suppose that we have a connected, weighted undirected graph $G = (V, E)$ such that every cut of G has a unique light edge crossing the cut. Show that G has exactly one minimum spanning tree.

Solution: From problem 2-1(a), we know that for every cut of G , the unique light edge crossing the cut must be part of every MST of G . Consider the set T of edges, containing the unique light edge crossing each cut C of G . By the previous part, we know that T is a subset of every MST.

Now we can show that T must be equal to every MST. Because there is exactly one edge of T crossing every cut of G , the graph (V, T) must have exactly one connected component: if there were two unconnected components V_1 and $V - V_1$, then the cut

$(V_1, V - V_1)$ would not be crossed by any edge of T , which contradicts the definition of T . In addition, T cannot contain any cycles: the heaviest edge on a cycle in T cannot be the unique lightest edge in any cut. Then T itself must be a spanning tree of G . Because T is a subset of all minimum spanning trees of G , it must be the unique MST of G , as any additional edge added to T would form a cycle, making it no longer be a tree.

In a graph with no duplicate edge weights, there cannot be any ties for the light edge crossing a cut. As a result, any graph with distinct edge weights will satisfy the requirements for the theorem in part (b), and will therefore have exactly one minimum spanning tree.

Problem 2-2. Shortest Paths and Minimum Spanning Trees

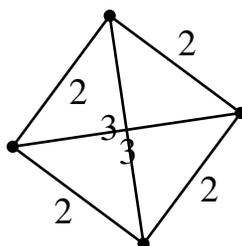
In this problem, you will examine the relationship between minimum spanning trees and shortest path trees. As a reminder, given a weighted undirected graph $G = (V, E)$ with edge weights w , the shortest path tree rooted at $s \in V$ is a subgraph $G' = (V', E')$ of G such that:

1. The vertex set $V' \subseteq V$ is the set of nodes in G reachable from s .
2. The graph G' is a tree, with $|E'| = |V'| - 1$.
3. For any $u \in V'$, the distance from s to u in G' is the same as the distance from s to u in G .

Note that as with minimum spanning trees, there is more than one shortest path tree per graph. In addition to the variation introduced by the choice of root, it's possible to get different shortest path trees even for the same root vertex.

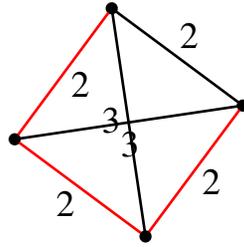
- (a) Given any connected undirected graph G with positive edge weights w , does there always exist a shortest path tree S such that S is a minimum spanning tree of G ? Prove your answer.

Solution: Here is a counterexample:

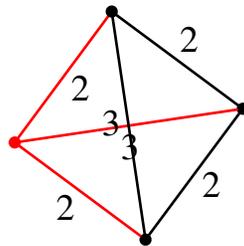


In the above graph, the outside edges each have a weight of 2, and the diagonal inner edges each have a weight of 3.

The minimum spanning trees of this element are any three of the weight-2 edges, as highlighted in red below:



The shortest path tree from each node contains just the edges containing that node, as shown below for the node highlighted in red:



- (b) Does there exist some connected undirected graph G with positive edge weights w such that G has a shortest path tree S and a minimum spanning tree T that do not share any edges? Prove your answer.

Solution: Assuming that G has more than one vertex, there is no such graph. Given any source vertex s , consider the light edges leaving s , defined as the edges (s, v) of minimum weight (where v is some neighbor of s).

Lemma 1 *Every MST of G must contain at least one of these light edges.*

PROOF. Suppose there is some MST T containing none of the light edges leaving s . Consider any light edge (s, u) . There is some path from s to u in T , and therefore adding (s, u) to T forms a cycle. There must be some other edge (s, v) in the cycle such that $w(s, u) < w(s, v)$, as we have assumed that there are no light edges leaving s contained in T . Removing (s, v) from T and replacing it with (s, u) forms a spanning tree of smaller total weight than T , contradicting our assumption that T is an MST.

Lemma 2 *All of the light edges leaving s must be contained in all shortest path trees rooted at s .*

PROOF. If (s, u) is a light edge leaving s , then because G has positive edge weights, the single edge (s, u) must be the unique shortest path from s to u . Any other path from s to u must go through some edge (s, v) , and by definition we know that $w(s, v) \geq w(s, u)$; therefore, all other paths from s to u must have strictly greater weight than the single edge (s, u) .

These two lemmas together show that, for any source node s , all shortest path trees from s must share some edge with all MSTs of G , so there is no shortest path tree S and MST T that share no edges.

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