

Problem Set 2

This problem set is due **at 9:00pm on Wednesday, February 22, 2012.**

Both exercises and problems should be solved, but *only the problems* should be turned in. Exercises are intended to help you master the course material. Even though you should not turn in the exercise solutions, you are responsible for material covered by the exercises.

Mark the top of the first page of your solution with your name, the course number, the problem number, your recitation section, the date, and the names of any students with whom you collaborated. The homework template (L^AT_EX) is available on the course website.

You will often be called upon to “give an algorithm” to solve a certain problem. Your write-up should take the form of a short essay. A topic paragraph should summarize the problem you are solving and what your results are. The body of the essay should provide the following:

1. A description of the algorithm in English and, if helpful, pseudo-code.
2. At least one worked example or diagram to show more precisely how your algorithm works.
3. A proof (or indication) of the correctness of the algorithm.
4. An analysis of the running time of the algorithm.

Remember, your goal is to communicate. Full credit will be given only to correct solutions *that are described clearly*. Convolved and opaque descriptions will receive lower marks.

Exercise 2-1. Do Exercise 21.1-2 on page 564 of CLRS.

Exercise 2-2. Do Exercise 23.1-2 on page 629 of CLRS.

Exercise 2-3. Do Exercise 23.1-5 on page 629 of CLRS.

Exercise 2-4. Do Exercise 23.1-10 on page 630 of CLRS.

Exercise 2-5. Do Exercise 23.1-11 on page 630 of CLRS.

Exercise 2-6. Do Exercise 23.2-4 on page 637 of CLRS.

Exercise 2-7. Do Exercise 23.2-8 on page 637 of CLRS.

Exercise 2-8. Do Problem 23-4 on page 641 of CLRS.

Problem 2-1. Unique Minimum Spanning Trees

In Lecture 3, we saw the claim that any weighted undirected graph with distinct edge weights has exactly one minimum spanning tree. In this problem, your goal will be to show that this claim is true by proving a more general theorem.

Let $G = (V, E)$ be a connected, weighted undirected graph whose edges weight may or may not be distinct. Given a cut $(S, V - S)$ of S , recall that an edge $(u, v) \in E$ is said to cross the cut if exactly one of u and v is in S . An edge (u, v) is a light edge crossing the cut $(S, V - S)$ if its weight is the smallest of any edge crossing $(S, V - S)$. There can be more than one light edge crossing a single cut in the case of ties. An edge (u, v) is a unique light edge if there are no ties.

- (a) Show that if an edge (u, v) is the unique light edge crossing some cut of the connected, weighted undirected graph G , then (u, v) must be included in *all* minimum spanning trees of G .
- (b) Suppose that we have a connected, weighted undirected graph $G = (V, E)$ such that every cut of G has a unique light edge crossing the cut. Show that G has exactly one minimum spanning tree.

In a graph with no duplicate edge weights, there cannot be any ties for the light edge crossing a cut. As a result, any graph with distinct edge weights will satisfy the requirements for the theorem in part (b), and will therefore have exactly one minimum spanning tree.

Problem 2-2. Shortest Paths and Minimum Spanning Trees

In this problem, you will examine the relationship between minimum spanning trees and shortest path trees. As a reminder, given a weighted undirected graph $G = (V, E)$ with edge weights w , the shortest path tree rooted at $s \in V$ is a subgraph $G' = (V', E')$ of G such that:

1. The vertex set $V' \subseteq V$ is the set of nodes in G reachable from s .
2. The graph G' is a tree, with $|E'| = |V'| - 1$.
3. For any $u \in V'$, the distance from s to u in G' is the same as the distance from s to u in G .

Note that as with minimum spanning trees, there is more than one shortest path tree per graph. In addition to the variation introduced by the choice of root, it's possible to get different shortest path trees even for the same root vertex.

- (a) Given any connected undirected graph G with positive edge weights w , does there always exist a shortest path tree S such that S is a minimum spanning tree of G ? Prove your answer.
- (b) Does there exist some connected undirected graph G with positive edge weights w such that G has a shortest path tree S and a minimum spanning tree T that do not share any edges? Prove your answer.

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