6.045: Automata, Computability, and Complexity
Or, Great Ideas in Theoretical Computer Science
Spring, 2010

Class 3

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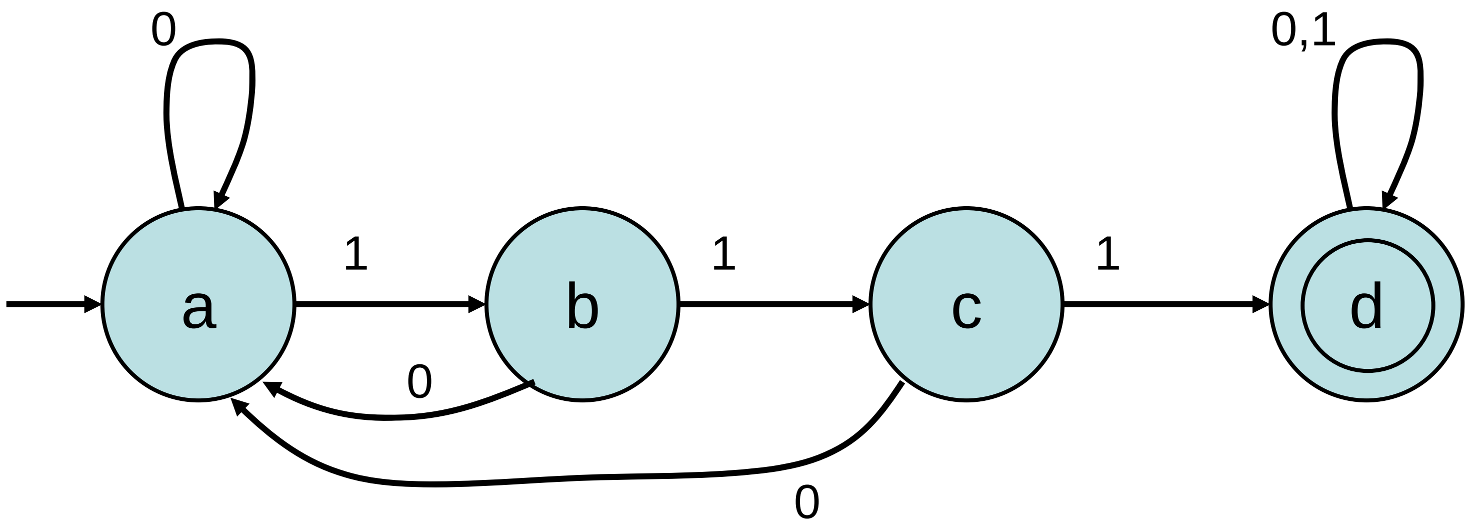
Today

- Finite Automata (FAs)
  - Our third machine model, after circuits and decision trees.
- Designed to:
  - Accept some strings of symbols.
  - Recognize a language, which is the set of strings it accepts.
- FA takes as its input a string of any length.
  - One machine for all lengths.
  - Circuits and decision trees use a different machine for each length.
- Today's topics:
  - Finite Automata and the languages they recognize
  - Examples
  - Operations on languages
  - Closure of FA languages under various operations
  - Nondeterministic FAs
- Reading: Sipser, Section 1.1.
- Next: Sections 1.2, 1.3.

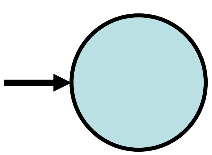
Finite Automata and the languages they recognize

Example 1

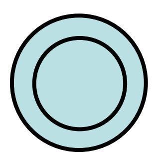
- An FA diagram, machine M



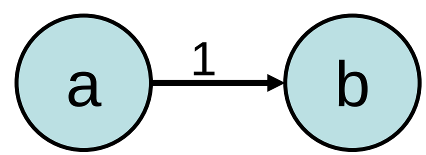
- Conventions:



Start state

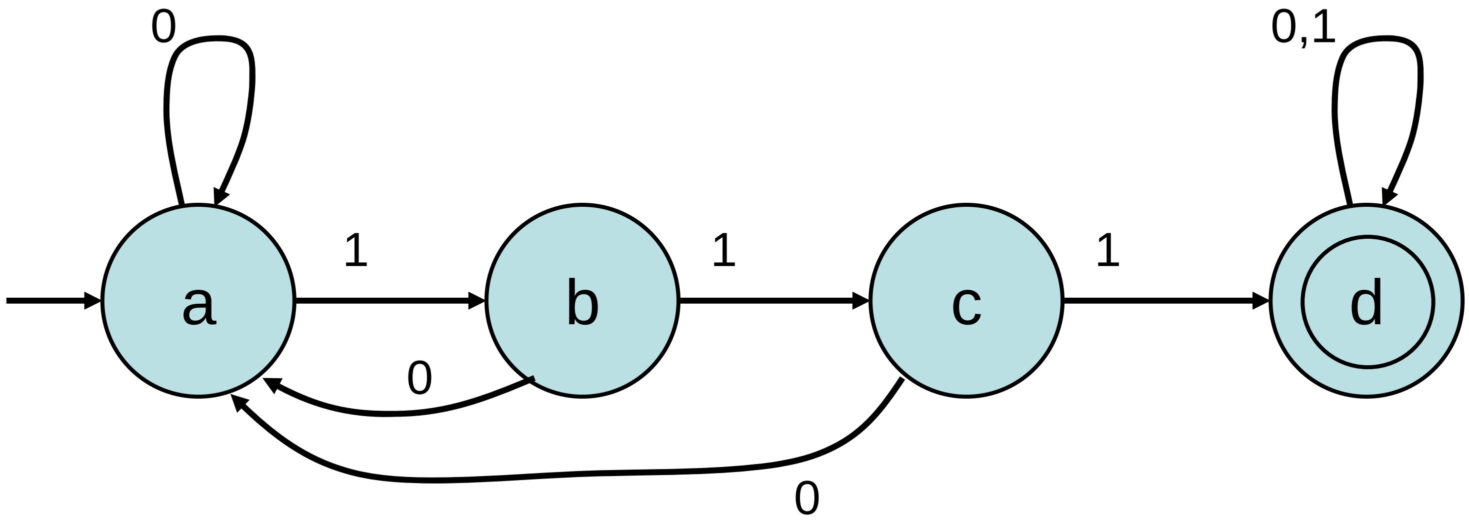


Accept state



Transition from a to b on input symbol 1.
Allow self-loops

Example 1



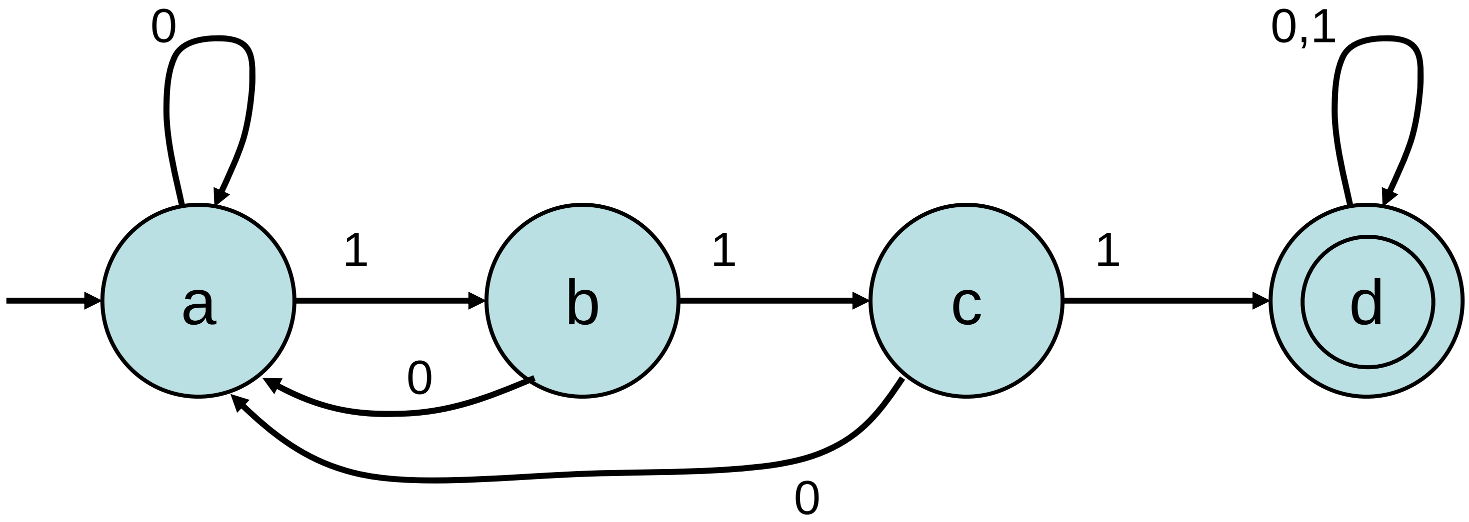
- Example computation:
  - Input word w: 1 0 1 1 0 1 1 1 0
  - States: a b a b c a b c d d
- We say that M accepts w, since w leads to d, an accepting state.

In general…

- A FA M accepts a word w if w causes M to follow a path from the start state to an accept state.
- Some terminology and notation:
  - Finite alphabet of symbols, usually called Σ.
  - In Example 1 (and often), $\Sigma = \{ 0,1 \}$.
  - String (word) over Σ: Finite sequence of symbols from Σ.
  - Length of w, $| w |$
  - ε, placeholder symbol for the empty string, $| \varepsilon | = 0$
  - Σ^*, the set of all finite strings of symbols in Σ
  - Concatenation of strings w and x, written $w \circ x$ or $w\,x$.
  - L(M), language recognized by M:

    $$\{ w \mid w \text{ is accepted by } M \}.$$
  - What is L(M) for Example 1?

Example 1



- What is L(M) for Example 1?
- { w ∈ { 0,1 }* | w contains 111 as a substring }
- Note: Substring refers to consecutive symbols.

Formal Definition of an FA

- An FA is a 5-tuple (Q, Σ, δ, q_0, F), where:
  - Q is a finite set of states,
  - Σ is a finite set (alphabet) of input symbols,
  - δ: $Q \times \Sigma \rightarrow Q$ is the transition function,

The arguments of δ are a state and an alphabet symbol.

The result is a state.

  - $q_0 \in Q$, is the start state, and
  - $F \subseteq Q$ is the set of accepting, or final states.

Example 1

- What is the 5-tuple $(Q, \Sigma, \delta, q_0, F)$?
- $Q = \{ a, b, c, d \}$
- $\Sigma = \{ 0, 1 \}$
- δ is given by the state diagram, or alternatively, by a table:

	0	1
a	a	b
b	a	c
c	a	d
d	d	d

- $q_0 = a$
- $F = \{ d \}$

Formal definition of computation

- Extend the definition of δ to input strings and states:

  $\delta^*: Q \times \Sigma^* \rightarrow Q$, state and string yield a state

  $\delta^*(q, w)$ = state that is reached by starting at q and following w.

- Defined recursively:

  $\delta^*(q, \varepsilon) = q$

  $\delta^*(q, w\,a) = \delta(\delta^*(q, w), a)$

  string symbol

- Or iteratively, compute $\delta^*(q, a_1\, a_2 \ldots a_k)$ by:

  $s := q$

  for $i = 1$ to k do $s := \delta(s, a_i)$

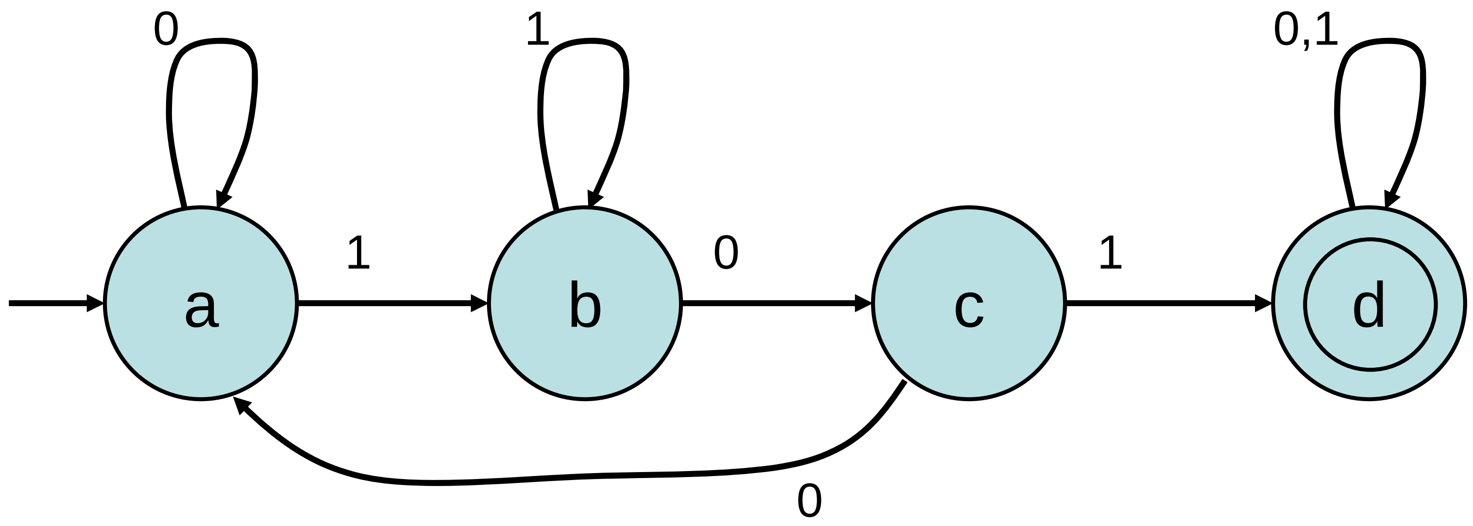
Formal definition of computation

- String w is accepted if $\delta^*(q_0, w) \in F$, that is, w leads from the start state to an accepting state.
- String w is rejected if it isn't accepted.
- A language is any set of strings over some alphabet.
- L(M), language recognized by finite automaton M = { w | w is accepted by M}.
- A language is regular, or FA-recognizable, if it is recognized by some finite automaton.

Examples of Finite Automata

Example 2

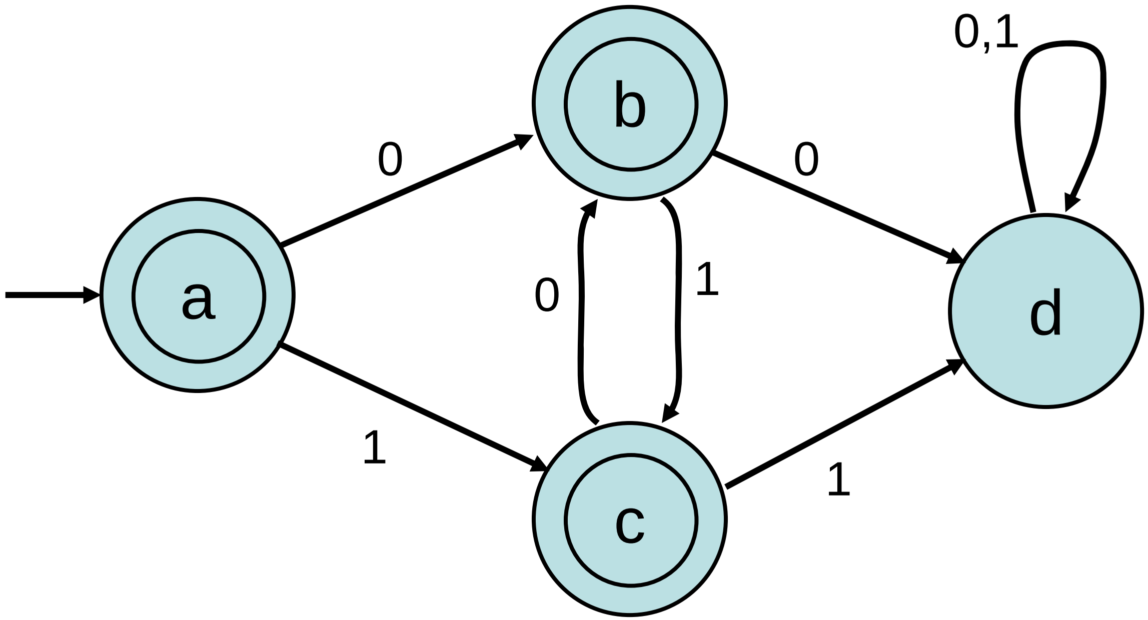
- Design an FA M with L(M) = { w ∈ { 0,1 }* | w contains 101 as a substring }.



- Failure from state b causes the machine to remain in state b.

Example 3

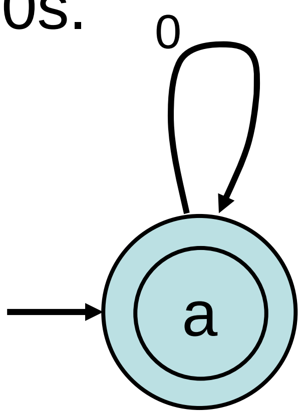
- L = { w ∈ { 0,1 }* | w doesn't contain either 00 or 11 as a substring }.



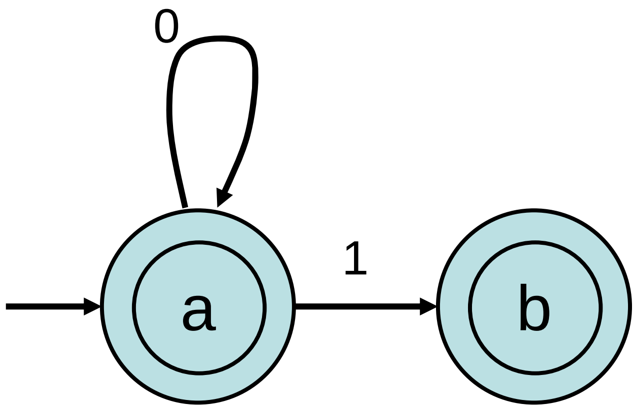
- State d is a trap state = a nonaccepting state that you can't leave.
- Sometimes we'll omit some arrows; by convention, they go to a trap state.

Example 4

- L = { w | all nonempty blocks of 1s in w have odd length }.
- E.g., ε, or 100111000011111, or any number of 0s.
- Initial 0s don't matter, so start with:



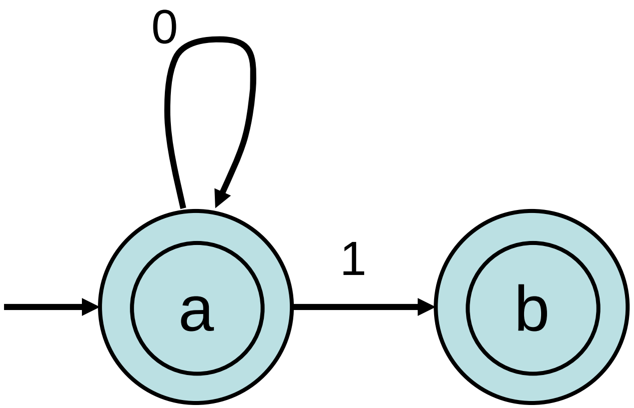
- Then 1 also leads to an accepting state, but it should be a different one, to "remember" that the string ends in one 1.

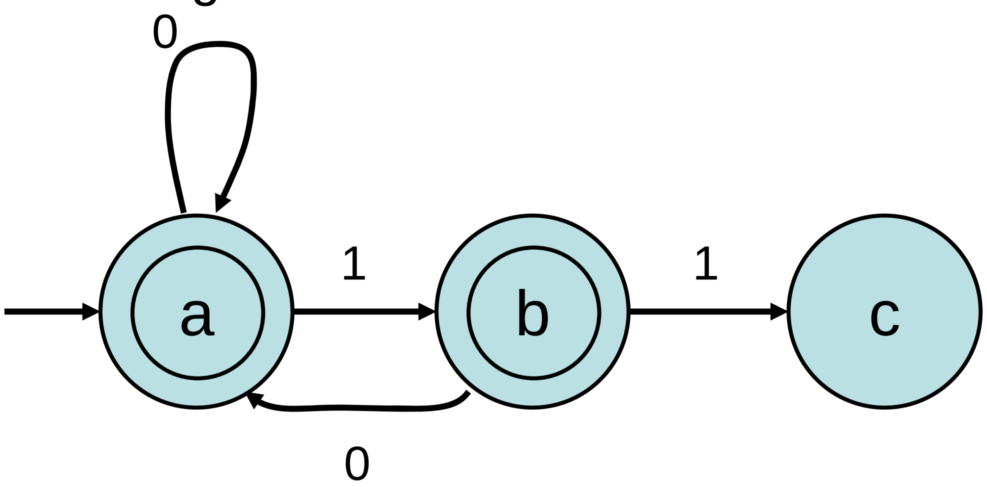


Example 4

- L = { w | all nonempty blocks of 1s in w have odd length }.

- From b:
  - 0 can return to a, which can represent either ε, or any string that is OK so far and ends with 0.
  - 1 should go to a new nonaccepting state, meaning "the string ends with two 1s".

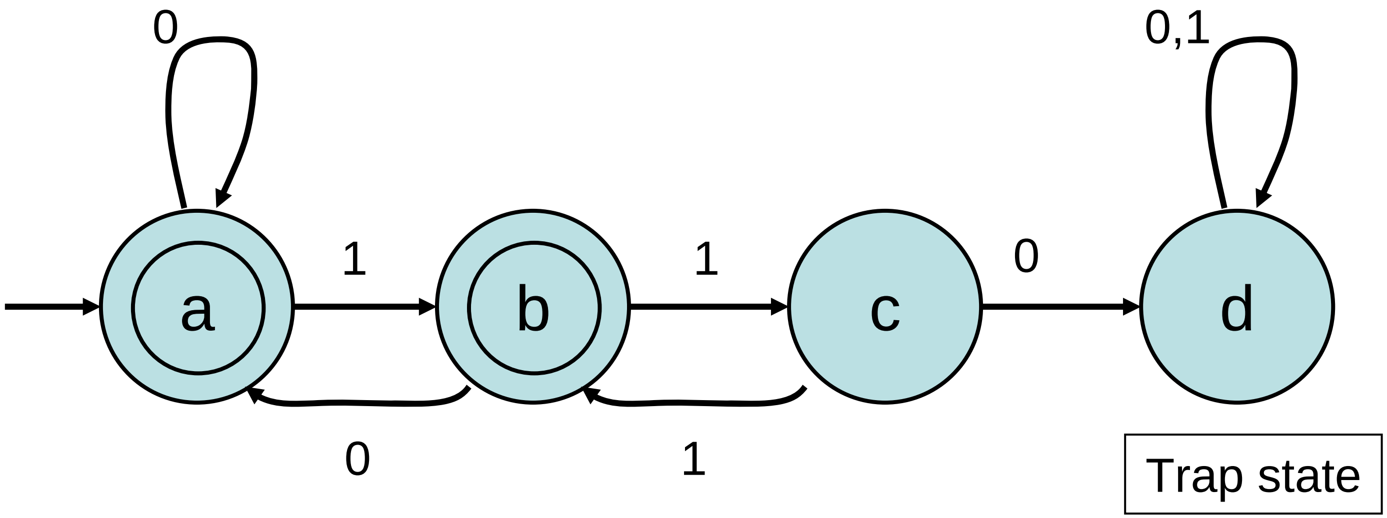




- Note: c isn't a trap state---we can accept some extensions.

Example 4

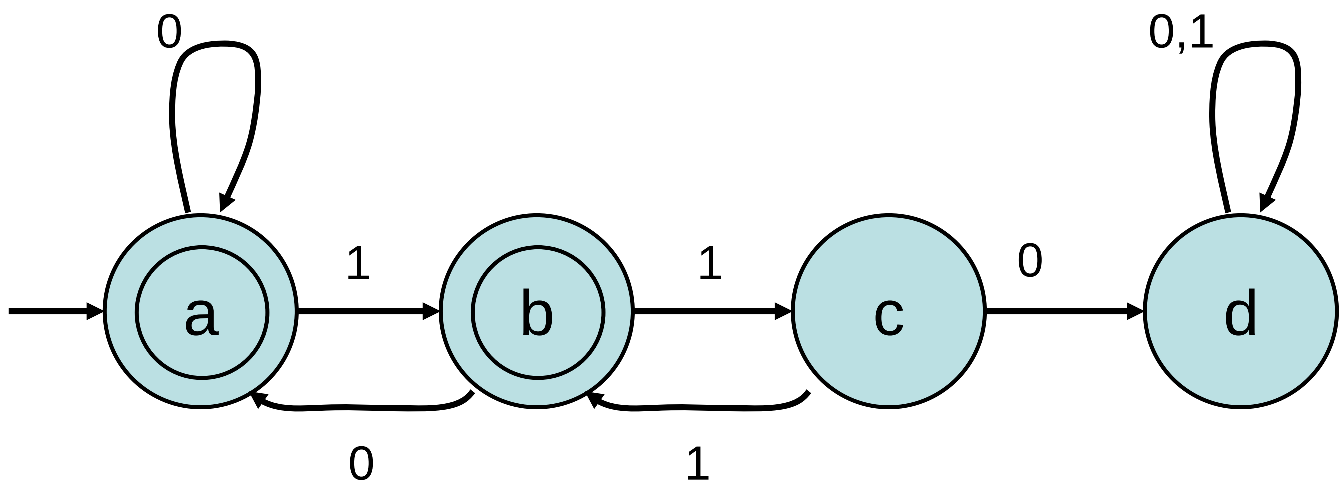
- L = { w | all nonempty blocks of 1s in w have odd length }.



- From c:
  - 1 can lead back to b, since future acceptance decisions are the same if the string so far ends with any odd number of 1s.
    - Reinterpret b as meaning “ends with an odd number of 1s”.
    - Reinterpret c as “ends with an even number of 1s”.
  - 0 means we must reject the current string and all extensions.

Example 4

- L = { w | all nonempty blocks of 1s in w have odd length }.



- Meanings of states (more precisely):

a: Either ε, or contains no bad block (even block of 1s followed by 0) so far and ends with 0.

b: No bad block so far, and ends with odd number of 1s.

c: No bad block so far, and ends with even number of 1s.

d: Contains a bad block.

Example 5

- L = EQ = { w | w contains an equal number of 0s and 1s }.
- No FA recognizes this language.
- Idea (not a proof):
  - Machine must "remember" how many 0s and 1s it has seen, or at least the difference between these numbers.
  - Since these numbers (and the difference) could be anything, there can't be enough states to keep track.
  - So the machine will sometimes get confused and give a wrong answer.
- We'll turn this into an actual proof next week.

Language Operations

Language operations

- Operations that can be used to construct languages from other languages.
- Recall: A language is any set of strings.
- Since languages are sets, we can use the usual set operations:
  - Union, $L_1 \cup L_2$
  - Intersection, $L_1 \cap L_2$
  - Complement, L^c
  - Set difference, $L_1 - L_2$
- We also have new operations defined especially for sets of strings:
  - Concatenation, $L_1 \circ L_2$ or just $L_1 L_2$
  - Star, L^*

Concatenation

- $L_1 \circ L_2 = \{ x\,y \mid x \in L_1 \text{ and } y \in L_2 \}$

  strings

  – Pick one string from each language and concatenate them.

- Example:

  $\Sigma = \{ 0, 1 \}$, $L_1 = \{ 0, 00 \}$, $L_2 = \{ 01, 001 \}$

  $L_1 \circ L_2 = \{ 001, 0001, 00001 \}$

- Notes:

  $| L_1 \circ L_2 | \leq | L_1 | \times | L_2 |$, not necessarily equal.

  $L \circ L$ does not mean $\{ x\,x \mid x \in L \}$, but rather, $\{ x\,y \mid x$ and y are both in $L \}$.

Concatenation

- $L_1 \circ L_2 = \{ x y \mid x \in L_1 \text{ and } y \in L_2 \}$

- Example:

  $\Sigma = \{ 0, 1 \}$, $L_1 = \{ 0, 00 \}$, $L_2 = \{ 01, 001 \}$

  $L_1 \circ L_2 = \{ 001, 0001, 00001 \}$

  $L_2 \circ L_2 = \{ 0101, 01001, 00101, 001001 \}$

- Example: $\varnothing \circ L$

  $\{ x y \mid x \in \varnothing \text{ and } y \in L \} = \varnothing$

- Example: $\{ \varepsilon \} \circ L$

  $\{ x y \mid x \in \{ \varepsilon \} \text{ and } y \in L \} = L$

Concatenation

- $L_1 \circ L_2 = \{ x\,y \mid x \in L_1 \text{ and } y \in L_2 \}$
- Write $L \circ L$ as L^2,

  $\underbrace{L \circ L \circ \ldots \circ L}_{\text{n of them}}$ as L^n, which is $\{ x_1\, x_2 \ldots x_n \mid$ all x's are in L $\}$

- Example: $L = \{ 0, 11 \}$

  $L^3 = \{ 000, 0011, 0110, 01111, 1100, 11011, 11110, 111111 \}$
- Example: $L = \{ 0, 00 \}$

  $L^3 = \{ 000, 0000, 00000, 000000 \}$
- Boundary cases:

  $L^1 = L$

  Define $L^0 = \{ \varepsilon \}$, for every L.
  - Implies that $L^0 L^n = \{ \varepsilon \} L^n = L^n$.
  - Special case of general rule $L^a L^b = L^{a+b}$.

The Star Operation

- $L^* = \{ x \mid x = y_1 y_2 \ldots y_k$ for some $k \geq 0$, where every y is in $L \}$

  $= L^0 \cup L^1 \cup L^2 \cup \ldots$

- Note: ε is in L^* for every L, since it's in L^0.
- Example: What is $\varnothing^*$?
  - Apply the definition:

$$\varnothing^* = \varnothing^0 \cup \varnothing^1 \cup \varnothing^2 \cup \ldots$$

The rest of these are just $\varnothing$.

This is $\{ \varepsilon \}$, by the convention that $L^0 = \{ \varepsilon \}$.

$$= \{ \varepsilon \}.$$

The Star Operation

- $L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$
- Example: What is { a }* ?
  - Apply the definition:

$$\{ a \}^* = \{ a \}^0 \cup \{ a \}^1 \cup \{ a \}^2 \cup \ldots$$
$$= \{ \varepsilon \} \cup \{ a \} \cup \{ a\ a \} \cup \ldots$$
$$= \{ \varepsilon, a, a\ a, a\ a\ a, \ldots \}$$

  - Abbreviate this to just a*.
  - Note this is not just one string, but a set of strings---any number of a's.

The Star Operation

- $L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$
- Example: What is Σ^* ?
  - We've already defined this to be the set of all finite strings over Σ.
  - But now it has a new formal definition:

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$$
$$= \{ \varepsilon \} \cup \{ \text{strings of length 1 over } \Sigma \}$$
$$\cup \{ \text{strings of length 2 over } \Sigma \}$$
$$\cup \ldots$$
$$= \{ \text{all finite strings over } \Sigma \}$$

  - Consistent.

Summary: Language Operations

- Set operations: Union, intersection, complement, set difference
- New language operations: Concatenation, star
- Regular operations:
  - Of these six operations, we identify three as regular operations: union, concatenation, star.
  - We'll revisit these next time, when we define regular expressions.

Closure of regular (FA-recognizable) languages under all six operations

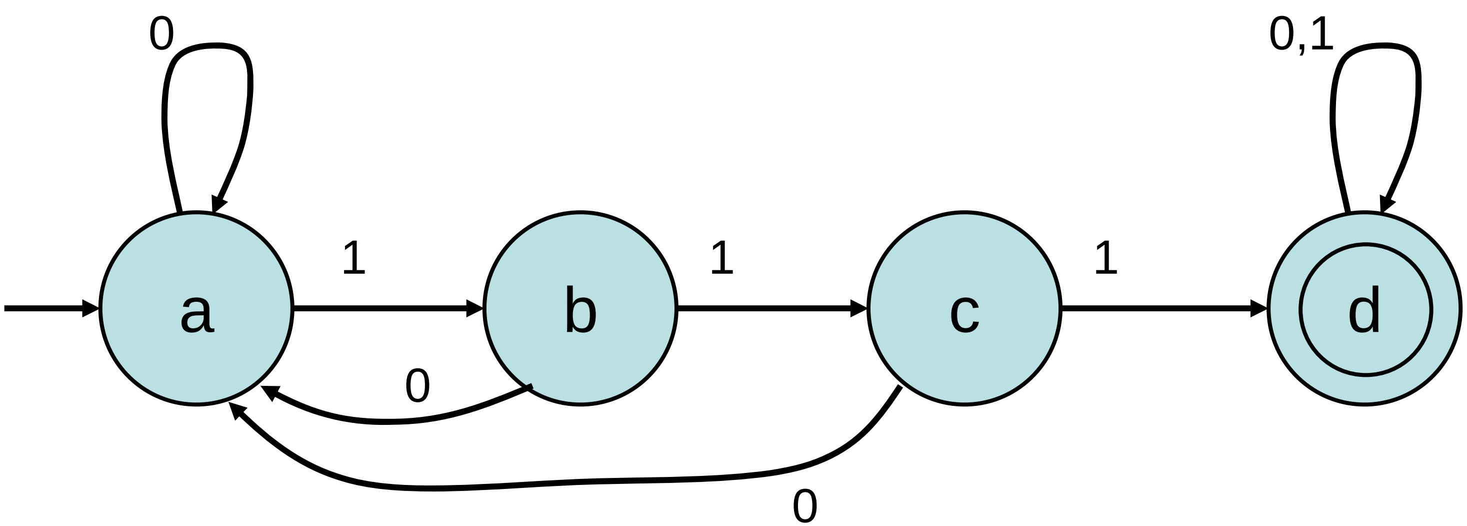
Closure under operations

- The set of FA-recognizable languages is closed under all six operations (union, intersection, complement, set difference, concatenation, star).
- This means: If we start with FA-recognizable languages and apply any of these operations, we get another FA-recognizable language (for a different FA).

- Theorem 1: FA-recognizable languages are closed under complement.
- Proof:
  - Start with a language L_1 over alphabet Σ, recognized by some FA, M_1.
  - Produce another FA, M_2, with $L(M_2) = \Sigma^* - L(M_1)$.
  - Just interchange accepting and non-accepting states.

Closure under complement

- Theorem 1: FA-recognizable languages are closed under complement.
- Proof: Interchange accepting and non-accepting states.
- Example: FA for { w | w does not contain 111 }
  - Start with FA for { w | w contains 111 }:



Closure under complement

- Theorem 1: FA-recognizable languages are closed under complement.
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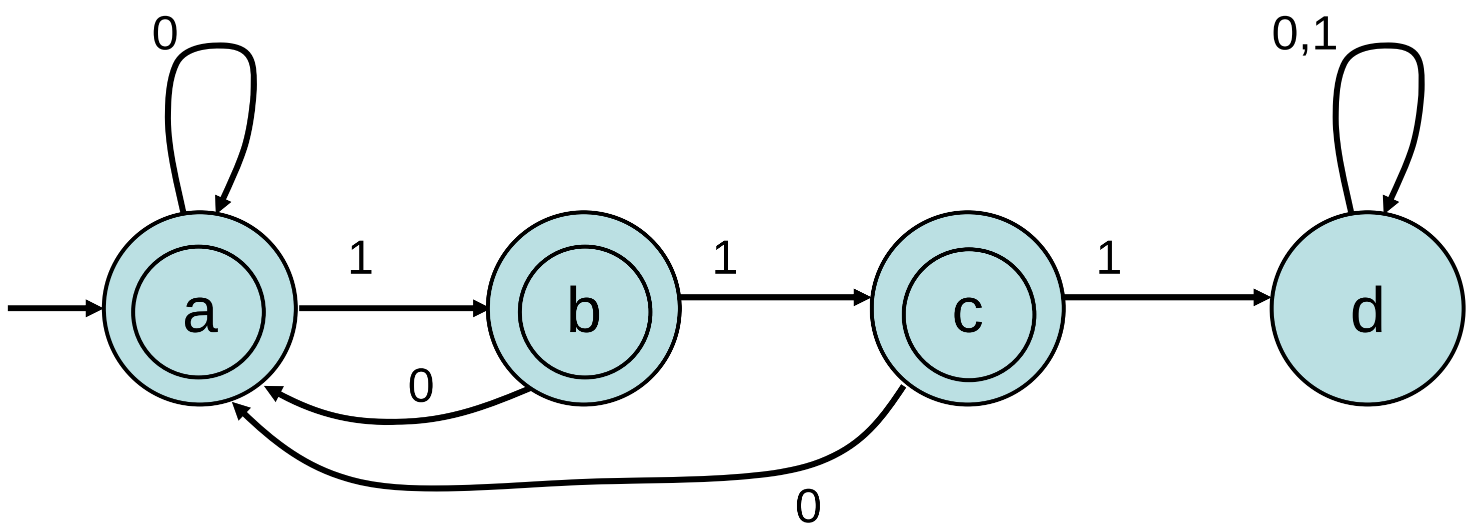


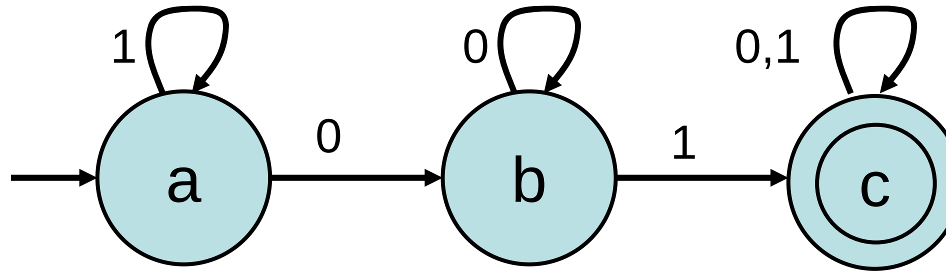
Closure under intersection

- Theorem 2: FA-recognizable languages are closed under intersection.
- Proof:
  - Start with FAs M_1 and M_2 for the same alphabet Σ.
  - Get another FA, M_3, with $L(M_3) = L(M_1) \cap L(M_2)$.
  - Idea: Run M_1 and M_2 "in parallel" on the same input. If both reach accepting states, accept.
  - Example:
    - $L(M_1)$: Contains substring 01.
    - $L(M_2)$: Odd number of 1s.
    - $L(M_3)$: Contains 01 and has an odd number of 1s.

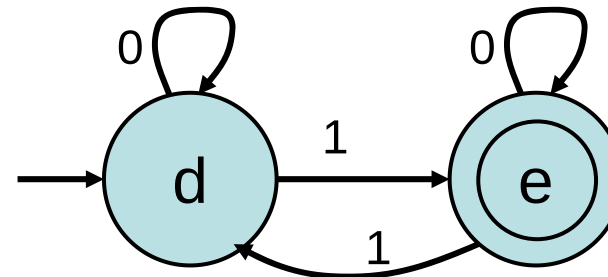
Closure under intersection

- Example:

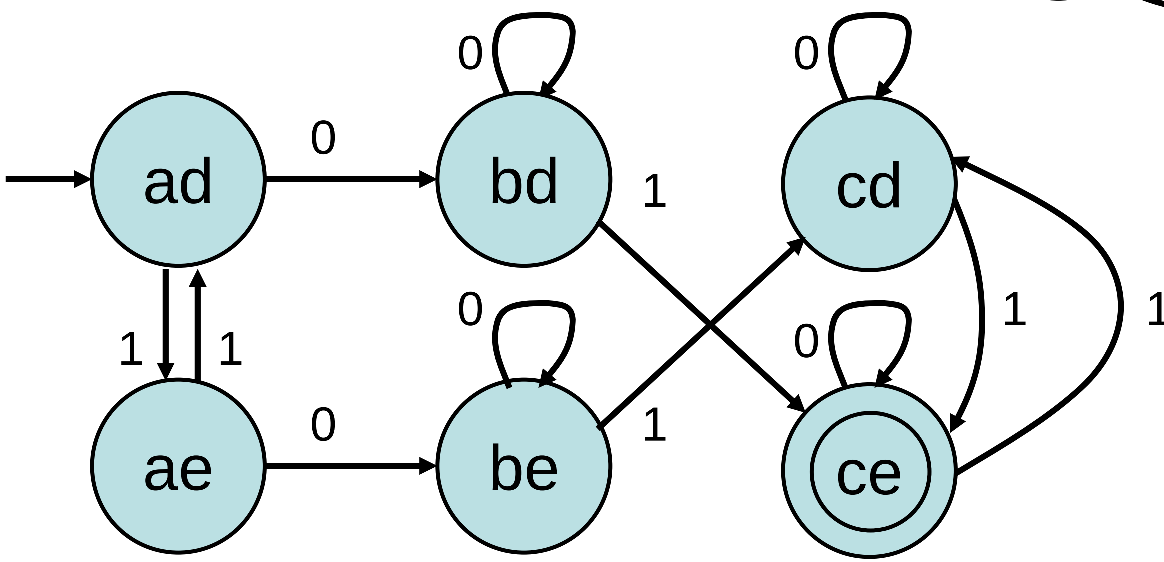
M_1: Substring 01



M_2: Odd number of 1s



M_3:



Closure under intersection, general rule

- Assume:
  - $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
  - $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- Define $M_3 = (Q_3, \Sigma, \delta_3, q_{03}, F_3)$, where
  - $Q_3 = Q_1 \times Q_2$
    - Cartesian product, $\{(q_1,q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}$
  - $\delta_3 ((q_1,q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$
  - $q_{03} = (q_{01}, q_{02})$
  - $F_3 = F_1 \times F_2 = \{ (q_1,q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2 \}$

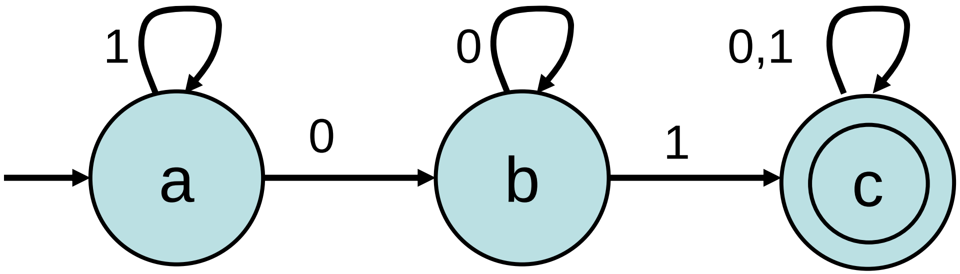
Closure under union

- Theorem 3: FA-recognizable languages are closed under union.
- Proof:
  - Similar to intersection.
  - Start with FAs M_1 and M_2 for the same alphabet Σ.
  - Get another FA, M_3, with $L(M_3) = L(M_1) \cup L(M_2)$.
  - Idea: Run M_1 and M_2 "in parallel" on the same input. If either reaches an accepting state, accept.
  - Example:
    - $L(M_1)$: Contains substring 01.
    - $L(M_2)$: Odd number of 1s.
    - $L(M_3)$: Contains 01 or has an odd number of 1s.

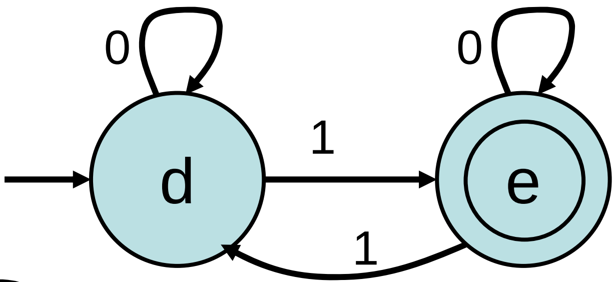
Closure under union

- Example:

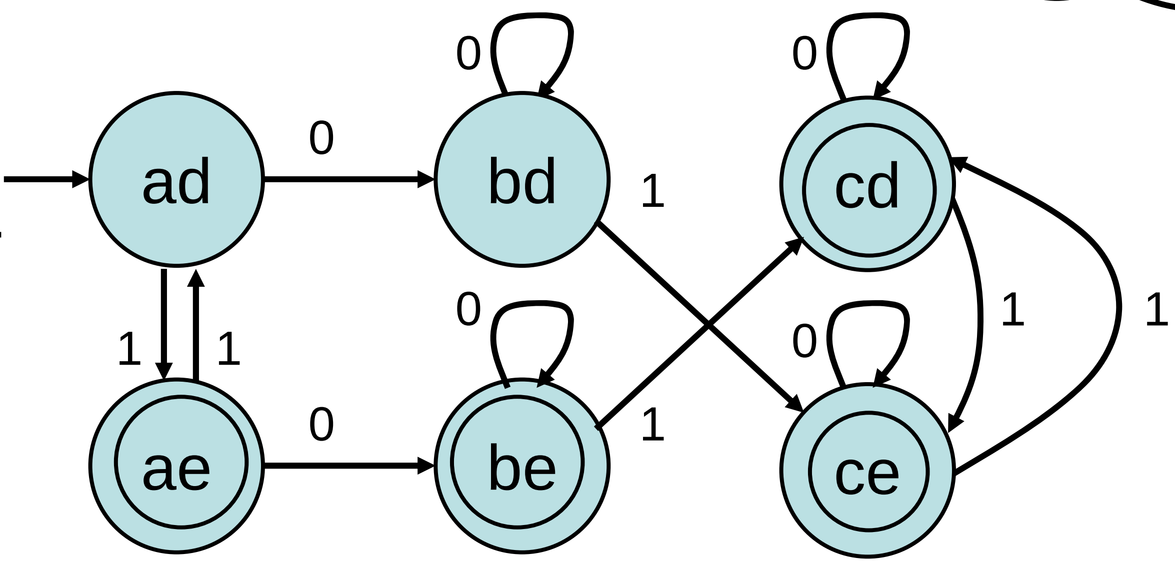
  M_1: Substring 01



  M_2: Odd number of 1s



  M_3: 1



Closure under union, general rule

- Assume:
  - $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$
  - $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- Define $M_3 = (Q_3, \Sigma, \delta_3, q_{03}, F_3)$, where
  - $Q_3 = Q_1 \times Q_2$
    - Cartesian product, $\{(q_1,q_2) \mid q_1 \in Q_1$ and $q_2 \in Q_2 \}$
  - $\delta_3 ((q_1,q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$
  - $q_{03} = (q_{01}, q_{02})$
  - $F_3 = \{ (q_1,q_2) \mid q_1 \in F_1$ or $q_2 \in F_2 \}$

Closure under set difference

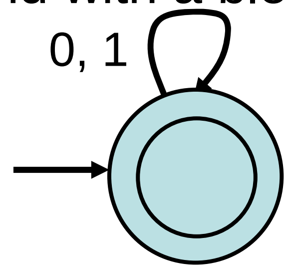
- Theorem 4: FA-recognizable languages are closed under set difference.
- Proof:
  - Similar proof to those for union and intersection.
  - Alternatively, since $L_1 – L_2$ is the same as $L_1 \cap (L_2)^c$, we can just apply Theorems 2 and 3.

Closure under concatenation

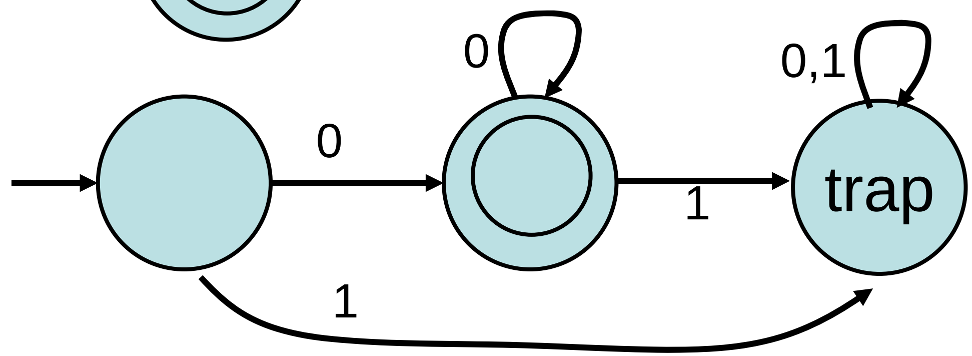
- Theorem 5: FA-recognizable languages are closed under concatenation.
- Proof:
  - Start with FAs M_1 and M_2 for the same alphabet Σ.
  - Get another FA, M_3, with $L(M_3) = L(M_1) \circ L(M_2)$, which is $\{ x_1 x_2 \mid x_1 \in L(M_1) \text{ and } x_2 \in L(M_2) \}$
  - Idea: ???
    - Attach accepting states of M_1 somehow to the start state of M_2.
    - But we have to be careful, since we don't know when we're done with the part of the string in $L(M_1)$---the string could go through accepting states of M_1 several times.

Closure under concatenation

- Theorem 5: FA-recognizable languages are closed under concatenation.
- Example:
  - Σ = { 0, 1}, $L_1 = \Sigma^*$, L_2 = {0} {0}* (just 0s, at least one).
  - $L_1 L_2$ = strings that end with a block of at least one 0
  - M_1:



  - M_2:



  - How to combine?
  - We seem to need to "guess" when to shift to M_2.
  - Leads to our next model, NFAs, which are FAs that can guess.

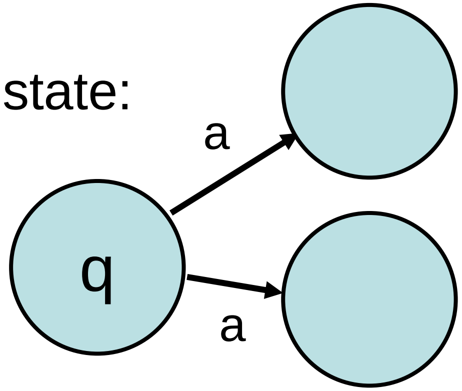
Closure under star

- Theorem 6: FA-recognizable languages are closed under star.
- Proof:
  - Start with FA M_1.
  - Get another FA, M_2, with $L(M_2) = L(M_1)^*$.
  - Same problems as for concatenation---need guessing.
  - …
  - We'll define NFAs next, then return to complete the proofs of Theorems 5 and 6.

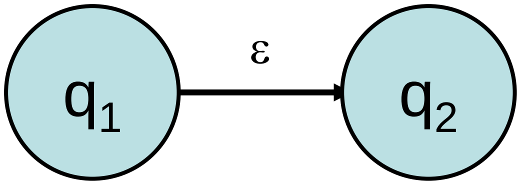
Nondeterministic Finite Automata

Nondeterministic Finite Automata

- Generalize FAs by adding nondeterminism, allowing several alternative computations on the same input string.
- Ordinary deterministic FAs follow one path on each input.
- Two changes:
  - Allow $\delta(q, a)$ to specify more than one successor state:



  - Add ε-transitions, transitions made "for free", without "consuming" any input symbols.



- Formally, combine these changes:

Formal Definition of an NFA

- An NFA is a 5-tuple (Q, Σ, δ, q_0, F), where:
  - Q is a finite set of states,
  - Σ is a finite set (alphabet) of input symbols,
  - $\delta: Q \times \Sigma_\varepsilon \to P(Q)$ is the transition function,

The arguments are a state and either an alphabet symbol or ε. Σ_ε means $\Sigma \cup \{\varepsilon\}$.

The result is a set of states.

  - $q_0 \in Q$, is the start state, and
  - $F \subseteq Q$ is the set of accepting, or final states.

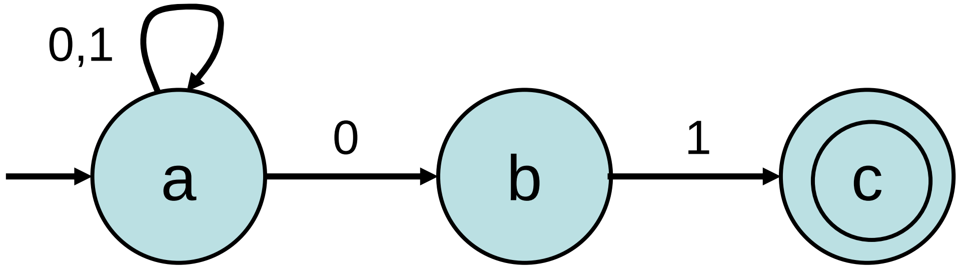
Formal Definition of an NFA

- An NFA is a 5-tuple (Q, Σ, δ, q_0, F), where:
  - Q is a finite set of states,
  - Σ is a finite set (alphabet) of input symbols,
  - $\delta: Q \times \Sigma_\varepsilon \rightarrow P(Q)$ is the transition function,
  - $q_0 \in Q$, is the start state, and
  - $F \subseteq Q$ is the set of accepting, or final states.
- How many states in P(Q)?

  $2^{|Q|}$
- Example: Q = { a, b, c }

  P(Q) = { $\varnothing$, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c} }

NFA Example 1



Q = { a, b, c }

Σ = { 0, 1 }

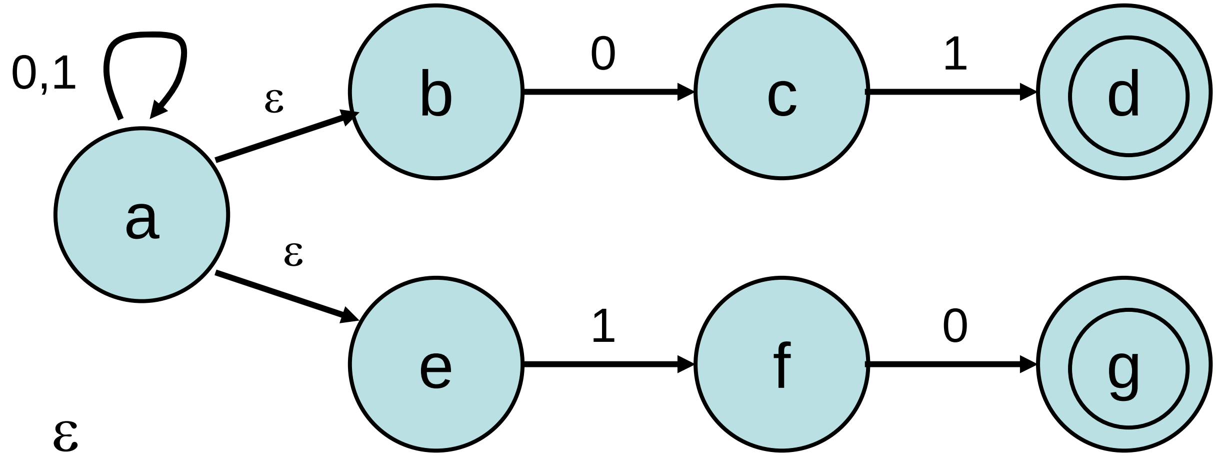
q_0 = a

F = { c }

δ:

	0	1	ε
a	{a,b}	{a}	∅
b	∅	{c}	∅
c	∅	∅	∅

NFA Example 2



	0	1	ε
a	{a}	{a}	{b,c}
b	{c}	∅	∅
c	∅	{d}	∅
d	∅	∅	∅
e	∅	{f}	∅
f	{g}	∅	∅
g	∅	∅	∅

Next time…

- NFAs and how they compute
- NFAs vs. FAs
- Closure of regular languages under languages operations, revisited
- Regular expressions
- Regular expressions denote FA-recognizable languages.
- Reading: Sipser, Sections 1.2, 1.3