

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Strong Induction

Albert R Meyer
February 24, 2012
lec 3F.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Strong Induction

Prove $P(0)$. Then prove $P(n+1)$ assuming all of

$P(0), P(1), \dots, P(n)$

(instead of just $P(n)$).

Conclude $\forall m.P(m)$

Albert R Meyer
February 24, 2012
lec 3F.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

available stamps:

5¢ 3¢

Thm: Get any amount $\geq 8\text{¢}$

By strong induction with hyp:
 $P(n) ::=$ can form $n + 8\text{¢}$.

Albert R Meyer
February 24, 2012
lec 3F.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

available stamps:

5¢ 3¢

Thm: Get any amount $\geq 8\text{¢}$

base case $P(0)$: make $0 + 8\text{¢}$

Albert R Meyer
February 24, 2012
lec 3F.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

available stamps:

5¢ 3¢

Thm: Get any amount $\geq 8\text{¢}$

inductive step:
 Assume $m+8\text{¢}$ for $n \geq m \geq 0$.

Albert R Meyer
February 24, 2012
lec 3F.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

available stamps:

5¢ 3¢

Thm: Get any amount $\geq 8\text{¢}$

inductive step:
 Assume all from 8 to $n+8\text{¢}$.

Albert R Meyer
February 24, 2012
lec 3F.6

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

available stamps:

5¢ 3¢

Thm: Get any amount $\geq 8¢$

inductive step:

Assume all from 8 to $n+8¢$.

Prove can get $n+9¢$, for $n \geq 0$

Albert R Meyer February 24, 2012 lec 3F.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

inductive step cases:

$n=0, 0+9¢ =$

$n=1, 1+9¢ =$

Albert R Meyer February 24, 2012 lec 3F.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

$n \geq 2$, so by hypothesis can get $(n-2)+8¢$

$(n-2)+8¢ + 3¢ = n+9¢$

O.K.

Albert R Meyer February 24, 2012 lec 3F.9

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Postage by Strong Induction

We conclude by strong induction that, using 3¢ and 5¢ stamps, $n + 8¢$ postage can be formed for all $n \geq 0$.

Albert R Meyer February 24, 2012 lec 3F.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Unstacking game

Start: a stack of boxes $a+b$

Move: split any stack into two of sizes $a, b > 0$

Scoring: $a \cdot b$ points

Keep moving: until stuck

Overall score: sum of move scores

Albert R Meyer February 24, 2012 lec 3F.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Analyzing the Stacking Game

Claim: Every way of unstacking n blocks gives the same score:

$$(n-1)+(n-2)+\dots+1 = \frac{n(n-1)}{2}$$

Albert R Meyer February 24, 2012 lec 3F.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Analyzing the Game

Claim: Starting with size n stack, final score will be

$$\frac{n(n-1)}{2}$$

Proof: by **Strong induction** with Claim(n) as hypothesis

Albert R Meyer

February 24, 2012

lec 3F.13

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction

Base case $n = 0$:

$$\text{score} = 0 = \frac{0(0-1)}{2}$$

Claim(0) is

Albert R Meyer

February 24, 2012

lec 3F.14

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction

Inductive step. Assume for stacks $\leq n$, and prove $C(n+1)$:

$$(n+1)\text{-stack score} = \frac{(n+1)n}{2}$$

Albert R Meyer

February 24, 2012

lec 3F.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction

Inductive step.

Case $n+1 = 1$. verify for 1-stack:

$$\text{score} = 0 = \frac{1(1-1)}{2}$$

C(1) is

Albert R Meyer

February 24, 2012

lec 3F.16

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction

Inductive step.

Case $n+1 > 1$. Split $n+1$ into an a -stack and b -stack, where $a + b = n + 1$.

$$(a + b)\text{-stack score} = ab + a\text{-stack score} + b\text{-stack score}$$

Albert R Meyer

February 24, 2012

lec 3F.17

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction

by **strong induction**:

$$a\text{-stack score} = \frac{a(a-1)}{2}$$

$$b\text{-stack score} = \frac{b(b-1)}{2}$$

Albert R Meyer

February 24, 2012

lec 3F.18

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction

total $(a + b)$ -stack score =

$$ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} =$$

$$\frac{(a+b)((a+b)-1)}{2} = \frac{(n+1)n}{2}$$

so $C(n+1)$ is 

We're done!



Albert R Meyer

February 24, 2012

lec 3F.19

MIT OpenCourseWare
<http://ocw.mit.edu>

6.042J / 18.062J Mathematics for Computer Science
Spring 2015

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.