



Recursive Functions



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Recursive Function

To define a function, f , on a recursively defined set R , define

- $f(b)$ explicitly for each base case $b \in R$
- $f(c(x))$ for each constructor, c , in terms of x and $f(x)$



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Recursive function on M

Def. $tree\text{-depth}(s)$ for $s \in M$

$$td(\lambda) ::= 0$$

$$td([s]t) ::= 1 + \max\{td(s), td(t)\}$$


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k^n — recursive function on \mathbb{N}

$$expt(k, 0) ::= 1$$

$$expt(k, n+1) ::= k \cdot expt(k, n)$$

--uses recursive def of \mathbb{N} :

- $0 \in \mathbb{N}$
- if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$



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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Recursive Functions

summary:

f : Data \rightarrow Values

$f(b)$ def'd directly for base b

$f(\text{cnstr}(x))$ def'd using $f(x)$, x



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Length versus Depth

Lemma: $|r| + 2 \leq 2^{\text{td}(r)+1}$

for all $r \in M$

Proof by Structural Induction

Base case: $[r = \lambda]$

$$|\lambda| + 2 = 0 + 2 = 2 = 2^{0+1} = 2^{\text{td}(\lambda)+1}$$

OK!



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Size versus Depth

Constructor case: $[r = [s]t]$

by ind. hypothesis:

$$|s| + 2 \leq 2^{\text{td}(s)+1}$$

$$|t| + 2 \leq 2^{\text{td}(t)+1}$$



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Size versus Depth

$$|r| + 2 = |[s]t| + 2 \quad \text{def. of } r$$

$$= (|s| + |t| + 2) + 2 \quad \text{def. of length}$$

$$= (|s| + 2) + (|t| + 2)$$

$$\leq 2^{\text{td}(s)+1} + 2^{\text{td}(t)+1} \quad \text{induction hyp.}$$

$$\leq 2^{\max(\text{td}(s), \text{td}(t))+1} + 2^{\max(\text{td}(s), \text{td}(t))+1}$$

$$= 2 \cdot 2^{\max(\text{td}(s), \text{td}(t))+1} \leq 2 \cdot 2^{\text{td}(r)} \quad \text{def. of } d(r)$$

$$= 2^{\text{td}(r)+1} \quad \text{QED!}$$



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positive powers of two

$2 \in \text{PP2}$
 if $x, y \in \text{PP2}$, then $x \cdot y \in \text{PP2}$
 $2, 2 \cdot 2, 4 \cdot 2, 4 \cdot 4, 4 \cdot 8, \dots$
 $2 \quad 4 \quad 8 \quad 16 \quad 32 \dots \in \text{PP2}$



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\log_2 of PP2

$\log_2(2) ::= 1$
 $\log_2(x \cdot y) ::= \log_2(x) + \log_2(y)$
 for $x, y \in \text{PP2}$
 $\log_2(4) = \log_2(2 \cdot 2) = 1 + 1 = 2$
 $\log_2(8) = \log_2(2 \cdot 4) = \log_2(2) + \log_2(4)$
 $= 1 + 2 = 3$



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loggy function on PP2

$\text{loggy}(2) ::= 1$
 $\text{loggy}(x \cdot y) ::= x + \text{loggy}(y)$
 for $x, y \in \text{PP2}$
 $\text{loggy}(4) = \text{loggy}(2 \cdot 2) = 2 + 1 = 3$
 $\text{loggy}(8) = \text{loggy}(2 \cdot 4) = 2 + \text{loggy}(4)$
 $= 2 + 3 = 5$
 $\text{loggy}(16) = \text{loggy}(8 \cdot 2) = 8 + \text{loggy}(2)$
 $= 8 + 1 = 9$



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loggy function on PP2

$\text{loggy}(16) = \text{loggy}(8 \cdot 2) = 9$
WAIT A SEC!
 $\text{loggy}(16) = \text{loggy}(2 \cdot 8)$
 $= 2 + \text{loggy}(8) = 2 + 5$
 $= 7$



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ambiguous constructors

The Problem: more than one way to construct elements of **PP2** from

$$\text{cnstrct}(x,y) = x \cdot y$$

$$16 = \text{cnstrct}(8,2) \text{ but also}$$

$$16 = \text{cnstrct}(2,8)$$

ambiguous



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ambiguous recursive defs

problem to watch out for:

recursive function on datum, e , is defined according to what constructor created e .

If 2 or more ways to construct e , then which definition to use?



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