Recursive Functions

Recursive Function

To define a function, f, on a recursively defined set R, define

- f(b) explicitly for each base case $b \in R$
- f(c(x)) for each constructor, c, in terms of x and f(x)

Recursive function on M

Def. tree-depth(s) for $s \in M$

$td(\lambda)$ ::= 0

td([s]t) ::=

1 + max{td(s), td(t)}

k^n — recursive function on $\mathbb{N}$

expt(k, 0) ::= 1

expt(k, n+1) ::= $k \cdot$ expt(k,n)

--uses recursive def of $\mathbb{N}$:

- $0 \in \mathbb{N}$
- if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$

Recursive Functions

summary:

f: Data → Values

$f(b)$ def'd directly for base b

$f(cnstr(x))$ def'd using $f(x)$, x

Length versus Depth

Lemma: $|r| + 2 \leq 2^{td(r)+1}$

for all $r \in M$

Proof by Structural Induction

Base case: $[r = \lambda]$

$|\lambda|+2 = 0+2 = 2 = 2^{0+1} = 2^{td(\lambda)+1}$

OK!

Size versus Depth

Constructor case: $[r = [s]t]$

by ind. hypothesis:

$$|s| + 2 \leq 2^{td(s)+1}$$

$$|t| + 2 \leq 2^{td(t)+1}$$

Size versus Depth

$|r|+2 = |[s]t| + 2$ def. of r

$= (|s|+|t| +2) + 2$ def. of length

$= (|s|+2)+(|t|+2)$

$\leq 2^{td(s)+1} + 2^{td(t)+1}$ induction hyp.

$\leq 2^{max(td(s),td(t))+1} + 2^{max(td(s),td(t))+1}$

$= 2 \cdot 2^{max(td(s),td(t))+1} \leq 2 \cdot 2^{td(r)}$ def. of $d(r)$

$= 2^{td(r)+1}$ QED!

positive powers of two

$2 \in PP2$

if $x,y \in PP2$, then $x \cdot y \in PP2$

$2,\ 2\cdot 2,\ 4\cdot 2,\ 4\cdot 4,\ 4\cdot 8, \ldots$

$2 \quad 4 \quad 8 \quad 16 \quad 32 \ldots \in PP2$

$\log_2$ of PP2

$\log_2(2) ::= 1$

$\log_2(x \cdot y) ::= \log_2(x) + \log_2(y)$ for $x,y \in PP2$

$\log_2(4) = \log_2(2 \cdot 2) = 1 + 1 = 2$

$\log_2(8) = \log_2(2 \cdot 4) = \log_2(2) + \log_2(4)$
$= 1 + 2 = 3$

loggy function on PP2

loggy(2) ::= 1

loggy($x \cdot y$) ::= x + loggy(y) for $x,y \in PP2$

loggy(4) = loggy($2 \cdot 2$) = 2 + 1 = 3

loggy(8) = loggy($2 \cdot 4$) = 2 + loggy(4)
= 2 + 3 = 5

loggy(16) = loggy($8 \cdot 2$) = 8 + loggy(2)
= 8 + 1 = 9

loggy function on PP2

loggy(16) = loggy($8 \cdot 2$) = 9

WAIT A SEC!:

loggy(16) = loggy($2 \cdot 8$)
= 2 + loggy(8) = 2 + 5
= 7

ambiguous constructors

The Problem: more than one way to construct elements of PP2 from

cnstrct(x,y) = x · y

16 = cnstrct(8,2) but also

16 = cnstrct(2,8)

ambiguous

ambiguous recursive defs

problem to watch out for:

recursive function on datum, e, is defined according to what constructor created e.

If 2 or more ways to construct e, then which definition to use?

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