

6	9	13	7
12		10	5
3	1	4	14
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# Induction

 Albert R Meyer February 24, 2012 lec 3F.1

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## The Idea of Induction

Color the integers  $\geq 0$   
 $0, 1, 2, 3, 4, 5, ?, \dots$   
 I tell you,  $0$  is red, & any int  
 next to a red integer is red,  
 then you know that  
 all the ints are red!

 Albert R Meyer February 24, 2012 lec 3F.2

6	9	13	7
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## The Idea of Induction

Color the integers  $\geq 0$   
 $0, 1, 2, 3, 4, 5, \dots$   
 I tell you,  $0$  is red, & any int  
 next to a red integer is red,  
 then you know that  
 all the ints are red!

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## Induction Rule

$R(0), R(0) \text{ IMPLIES } R(1), R(1) \text{ IMPLIES } R(2),$   
 $R(2) \text{ IMPLIES } R(3), \dots, R(n) \text{ IMPLIES } R(n+1), \dots$   


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 $R(0), R(1), R(2), \dots, R(n), \dots$

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6	9	13	7
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## Induction Rule

$$R(0), \forall n. R(n) \text{ IMPLIES } R(n+1)$$


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$$\forall m. R(m)$$


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6	9	13	7
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## Like Dominos...

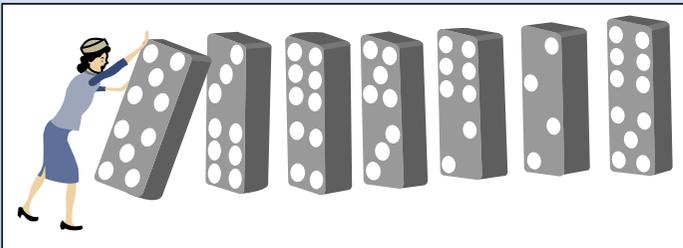


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6	9	13	7
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## Example Induction Proof

Let's prove:

$$1+r+r^2+\dots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for  $r \neq 1$ )


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## Example Induction Proof

Statements in magenta form a template for inductive proofs:

Proof: (by induction on  $n$ )

The induction hypothesis,  $P(n)$ , is:

$$1+r+r^2+\dots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for  $r \neq 1$ )


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### Example Induction Proof

Base Case ( $n = 0$ ):

$$\underbrace{1+r+r^2+\dots+r^0}_{1} = \overset{?}{\frac{r^{0+1}-1}{r-1}} = \frac{r-1}{r-1} = 1$$

OK!

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### Example Induction Proof

Inductive Step: Assume  $P(n)$  where  $n \geq 0$  and prove  $P(n+1)$ :

$$1+r+r^2+\dots+r^{n+1} = \frac{r^{(n+1)+1}-1}{r-1}$$

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### Example Induction Proof

Now from induction hypothesis  $P(n)$  we have

$$1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1}$$

so add  $r^{n+1}$  to both sides

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### Example Induction Proof

adding  $r^{n+1}$  to both sides,

$$(1+r+r^2+\dots+r^n) + r^{n+1} = \left(\frac{r^{n+1}-1}{r-1}\right) + r^{n+1}$$

This proves  $P(n+1)$  completing the proof by induction.

$$= \frac{r^{n+1}-1+r^{n+1}(r-1)}{r-1} = \frac{r^{(n+1)+1}-1}{r-1}$$

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## an aside: ellipsis

"..." is an **ellipsis**. Means you should **see a pattern**:

$$1 + r + r^2 + \dots + r^n = \sum_{i=0}^n r^i$$

Can lead to confusion ( $n = 0$ ?)  
**sum ( $\Sigma$ ) notation** more precise



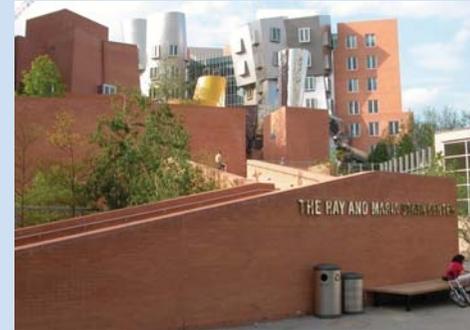
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6	9	13	7
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## The MIT Stata Center



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## Design Mockup: Stata Lobby

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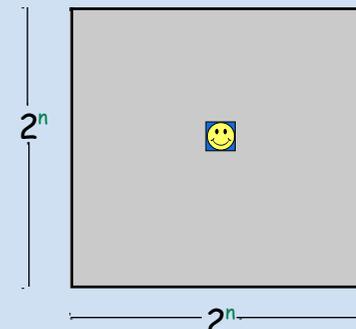
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6	9	13	7
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## Mockup: Plaza Outside Stata

Goal: Tile the plaza, except for  $1 \times 1$  square in the middle for Bill.



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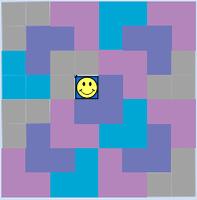
6	9	13	7
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## Plaza Outside Stata

Gehry specifies L-shaped tiles covering three squares:



For example, for 8 x 8 plaza might tile for Bill this way:




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## Plaza Outside Stata

Theorem: For any  $2^n \times 2^n$  plaza, we can make Bill and Frank happy.

Proof: (by induction on  $n$ )  
 $P(n) ::=$  can tile  $2^n \times 2^n$  with Bill in middle.

Base case: ( $n=0$ )

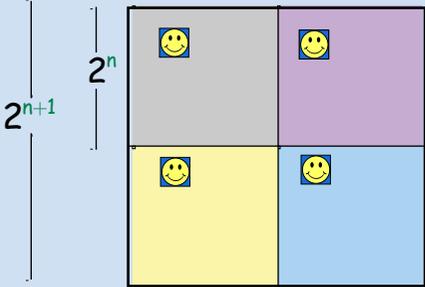
 (no tiles needed)


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6	9	13	7
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## Plaza Outside Stata

Induction step: assume can tile  $2^n \times 2^n$ , prove can tile  $2^{n+1} \times 2^{n+1}$

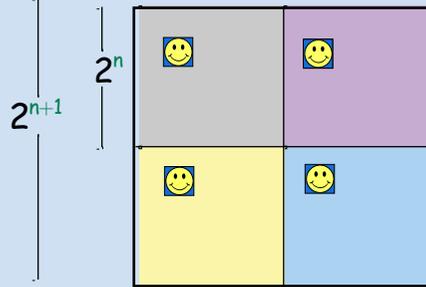



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## Plaza Outside Stata

Now what?...




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### Plaza Outside Stata

The fix:

prove something stronger  
—that we can find a tiling  
with Bill in any square.



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### Plaza Theorem

Theorem: For any  $2^n \times 2^n$  plaza, we  
can make Bill and Frank happy.

Proof: (by induction on  $n$ )  
revised induction hypothesis  $P(n) ::=$   
can tile with Bill anywhere

Base case: ( $n=0$ ) as before



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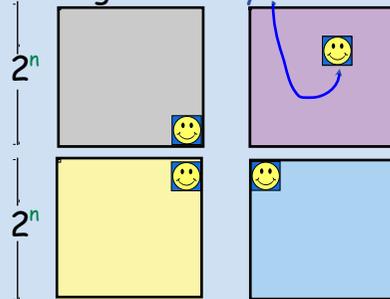
6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### Plaza Proof

Inductive step:

Assume we can get Bill anywhere in  $2^n \times 2^n$

Prove we can get Bill anywhere in  $2^{n+1} \times 2^{n+1}$



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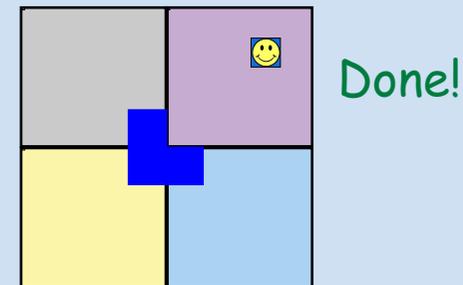
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6	9	13	7
12		10	5
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### Plaza Proof

Now group the squares together,  
and fill the center Bill's with a tile.



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## Recursive Procedure

**Note:** The induction proof implicitly defines a recursive procedure for tiling with Bill anywhere.



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