

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

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# Derived Variables

 Albert R Meyer February 27, 2013 derived-var.1

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

## Derived Variables

A *derived variable*,  $v$ , is a function assigning a "value" to each state:

$v: \text{States} \rightarrow \text{Values}$

If  $\text{Vals} = \mathbb{N}$ , say  $v$  is " $\mathbb{N}$ -valued" or "nonnegative-integer-valued"

 Albert R Meyer February 27, 2013 derived-var.2

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

## Derived Variables

Robot on the grid example:

States =  $\mathbb{N}^2$ . Define the sum-value,  $\sigma$ , of a state:

$\sigma(x,y) ::= x+y$

an  $\mathbb{N}$ -valued derived variable

 Albert R Meyer February 27, 2013 derived-var.3

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

## Derived Variables

Called *derived* to distinguish from *actual* variables that appear in a program.

For robot **Actual:**  $x, y$   
**Derived:**  $\sigma$

 Albert R Meyer February 27, 2013 derived-var.4

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

### Derived Variables

Another derived variable:

$$\pi ::= \sigma \pmod{2}$$

$\pi$  is  $\{0,1\}$ -valued

Albert R Meyer February 27, 2013 derived-var.5

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

### Derived Variables

For Fast Exp, have (actual) variable Z.

Proof of termination:

Z is strictly decreasing & natural number-valued

Albert R Meyer February 27, 2013 derived-var.6

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

### Derived Variables

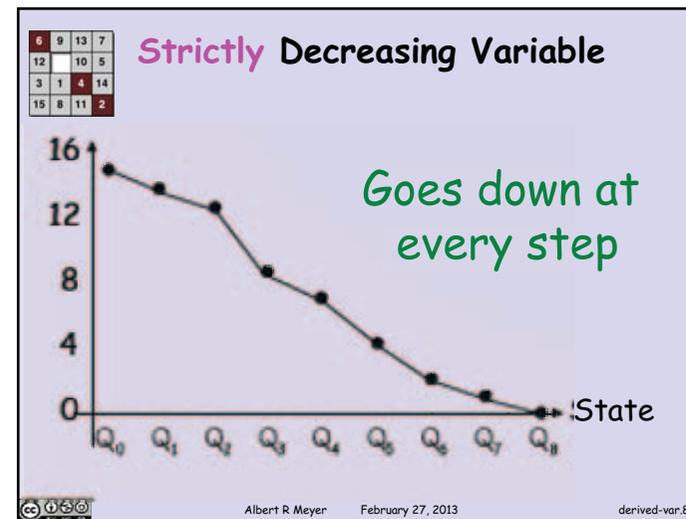
Termination followed by

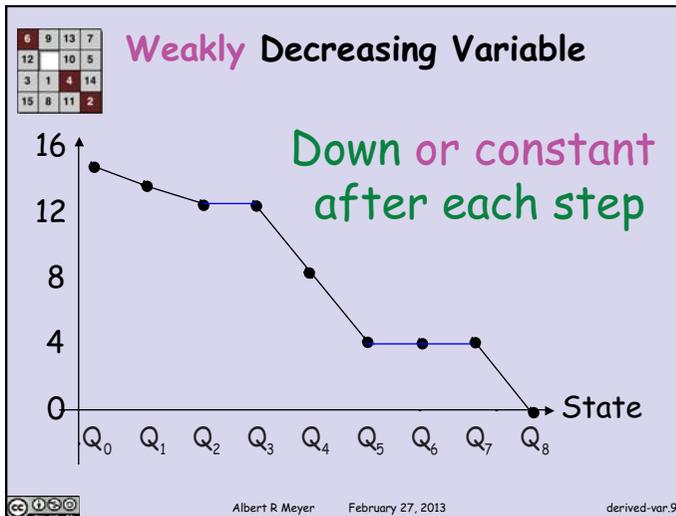
Well Ordering Principle:

Z must take a least value.

then the algorithm is stuck

Albert R Meyer February 27, 2013 derived-var.7





|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

### Diagonal Robot variables

□: up & down all over the place

neither increasing  
nor decreasing

π: is constant

both weakly increasing  
& weakly decreasing

Albert R Meyer February 27, 2013 derived-var.10

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

### Weakly Decreasing Variable

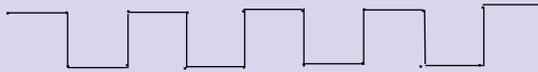
(We used to call weakly  
decreasing variables  
"nonincreasing"  
variables.)

Albert R Meyer February 27, 2013 derived-var.11

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

### Weakly Decreasing Variable

OK terminology but remember:  
nonincreasing is  
NOT SAME as "not increasing:"



Albert R Meyer February 27, 2013 derived-var.12

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

## Well ordered sets

*Def.* A set  $W$  of real numbers is **well ordered** iff it has **no infinite decreasing sequence**

$$w_0 > w_1 > w_2 > \dots > w_n > \dots$$

 Albert R Meyer February 27, 2013 derived-var.13

|    |   |    |    |
|----|---|----|----|
| 6  | 9 | 13 | 7  |
| 12 |   | 10 | 5  |
| 3  | 1 | 4  | 14 |
| 15 | 8 | 11 | 2  |

## Well ordered sets

Termination using WOP on  $\mathbb{N}$  generalizes to strictly decreasing variables whose values are in any **well ordered set**

 Albert R Meyer February 27, 2013 derived-var.14

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