Mathematics for Computer Science

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Set Theory: ZFC

Zermelo-Frankel Set Theory

Axioms of Zermelo-Frankel with the Choice axiom (ZFC) define the standard Theory of Sets

Some Axioms of Set Theory

Extensionality

x and y have the same elements

Some Axioms of Set Theory

Extensionality

$\forall x[x \in y \text{ IFF } x \in z]$

iff

x and y are members of the same sets

Some Axioms of Set Theory

Extensionality

$\forall x[x \in y \text{ IFF } x \in z]$

iff

$\forall x[y \in x \text{ IFF } z \in x]$

Some Axioms of Set Theory

Power set

Every set has a power set

$\forall x \exists p \forall s. s \subseteq x \text{ IFF } s \in p$

Some Axioms of Set Theory

Comprehension

If S is a set, and $P(x)$ is a predicate of set theory, then

$$\{x \in s \mid P(x)\}$$

is a set

Sets are Well Founded

According to ZF, the elements of a set have to be "simpler" than the set itself. In particular,

no set is a member of itself, or a member of a member…

Sets are Well Founded

Def: x is $\in$-minimal in y

x is in y, but no element of x is in y

Sets are Well Founded

Def: x is $\in$-minimal in y

$x \in y$ AND

$[\forall z. z \in x$ IMPLIES $z \notin y]$

Some Axioms of Set Theory

Foundation

Every nonempty set has an $\in$-minimal element

Some Axioms of Set Theory

Foundation

Every nonempty set has an $\in$-minimal element

$\forall x.[x \neq \emptyset$ IMPLIES

$\exists y. y$ is $\in$-minimal in $x]$

$S \notin S$

Let $R ::= \{S\}$.

$S \notin S$

Let $R ::= \{S\}$. If $S \in S$, then R has no $\in$-minimal element.
If it exists, it must be S,
but $S \in R$ and $S \in S$,
so S is not $\in$-minimal in R.

Zermelo-Frankel Set Theory

$S \notin S$ implies that

(1) the collection of all sets is not a set, and

(2) $W = \{s \in \text{Sets} \mid s \notin s\}$

Zermelo-Frankel Set Theory

$S \notin S$ implies that

(1) the collection of all sets is not a set, and

(2) $W = \{s \in \text{Sets} \mid s \notin s\}$ is the collection of all sets -- which is why it's not a set.