

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

Set Theory: ZFC



Albert R Meyer, March 4, 2015

ZF.1

6	9	13	7
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Zermelo-Frankel Set Theory

Axioms of Zermelo-Frankel
with the Choice axiom
(ZFC) define the standard
Theory of Sets



Albert R Meyer, March 4, 2015

ZF.2

6	9	13	7
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Some Axioms of Set Theory

Extensionality

x and y have the same elements



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ZF.3

6	9	13	7
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Some Axioms of Set Theory

Extensionality

$$\forall x [x \in y \text{ IFF } x \in z]$$

iff

x and y are members of the
same sets



Albert R Meyer, March 4, 2015

ZF.4

6	9	13	7
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Some Axioms of Set Theory

Extensionality

$$\forall x[x \in y \text{ IFF } x \in z]$$

iff

$$\forall x[y \in x \text{ IFF } z \in x]$$



Albert R Meyer, March 4, 2015

ZF.5

6	9	13	7
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Some Axioms of Set Theory

Power set

Every set has a power set

$$\forall x \exists p \forall s. s \subseteq x \text{ IFF } s \in p$$



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ZF.6

6	9	13	7
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Some Axioms of Set Theory

Comprehension

If S is a set, and $P(x)$ is a predicate of set theory, then

$$\{x \in S \mid P(x)\}$$

is a set



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ZF.7

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Sets are Well Founded

According to ZF, the elements of a set have to be "simpler" than the set itself. In particular,

no set is a member of itself, or a member of a member...



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ZF.8

6	9	13	7
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Sets are Well Founded

Def: x is \in -minimal in y

x is in y , but no element
of x is in y



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ZF.9

6	9	13	7
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Sets are Well Founded

Def: x is \in -minimal in y

$x \in y$ AND

$[\forall z. z \in x \text{ IMPLIES } z \notin y]$



Albert R Meyer, March 4, 2015

ZF.10

6	9	13	7
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Some Axioms of Set Theory

Foundation

Every nonempty set has
an \in -minimal element



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ZF.11

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Some Axioms of Set Theory

Foundation

Every nonempty set has
an \in -minimal element

$\forall x. [x \neq \emptyset \text{ IMPLIES}$

$\exists y. y \text{ is } \in\text{-minimal in } x]$



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ZF.12

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$$S \notin S$$

Let $R ::= \{S\}$.



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$$S \notin S$$

Let $R ::= \{S\}$. If $S \in S$, then R has no \in -minimal element. If it exists, it must be S , but $S \in R$ and $S \in S$, so S is not \in -minimal in R .



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Zermelo-Frankel Set Theory

$S \notin S$ implies that

- (1) the collection of all sets is not a set, and
- (2) $W = \{s \in \text{Sets} \mid s \notin s\}$



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Zermelo-Frankel Set Theory

$S \notin S$ implies that

- (1) the collection of all sets is not a set, and
- (2) $W = \{s \in \text{Sets} \mid s \notin s\}$ is the collection of all sets -- which is why it's not a set.



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