

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
 MIT 6.042J/18.062J

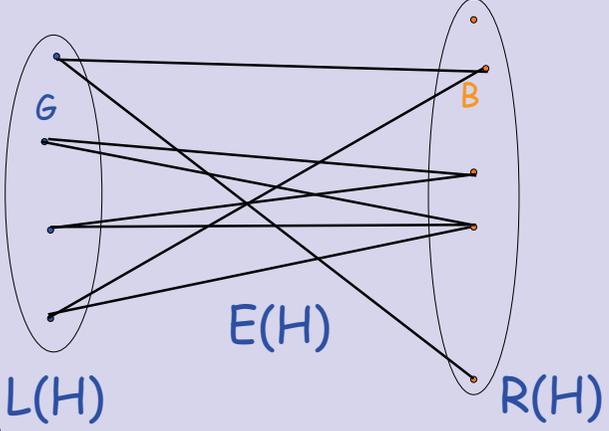
Hall's Theorem

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Hall.1

6	9	13	7
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15	8	11	2

Hall graph H



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Hall.2

6	9	13	7
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Hall graph H

A match is a
 total injective function
 $m: G \rightarrow B$
 that follows edges

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Hall.3

6	9	13	7
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Hall graph H

A match is a
 total injective function
 $m: G \rightarrow B$
 $g - m(g) \in E$

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Hall.4

6	9	13	7
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Hall graph H

A match is a
total injective function

$$m: G \rightarrow B$$

$$\text{graph}(m) \subseteq E$$



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Hall.5

6	9	13	7
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Hall's Theorem

Hall's condition

If $|S| \leq |E(S)|$ for all
sets of girls, S ,
then there is a match.



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Hall.6

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How to verify no bottlenecks?

fairly efficient matching
procedure is known

(explained in algorithms subjects)

...but there is a special
situation that ensures a
match...



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Hall.7

6	9	13	7
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How to verify no bottlenecks?

If every girl likes $\geq d$ boys,
and every boy likes $\leq d$ girls,
then no bottlenecks.

a degree-constrained
Hall graph



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Hall.8



How to verify no bottlenecks?

If every girl likes $\geq d$ boys,
and every boy likes $\leq d$ girls,
then no bottlenecks.

proof:

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How to verify no bottlenecks?

If every girl likes $\geq d$ boys,
and every boy likes $\leq d$ girls,
then no bottlenecks.

proof: say set S of girls has e
incident edges:

$$d \cdot |S| \leq e \leq d \cdot |E(S)|$$

so $|S| \leq |E(S)|$

no bottleneck

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Proof of Hall's Theorem

Suppose no bottlenecks.

Lemma: No bottlenecks
within any set S of girls.

obviously

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Proof of Hall's Theorem

Suppose no bottlenecks.

Lemma: If S a set of girls with
 $|S| = |E(S)|$,
then no bottlenecks between
 \overline{S} and $\overline{E(S)}$ either

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bottleneck between \overline{S} & $\overline{E(S)}$?

then $T \cup S$
is a bottleneck ✗

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Proof of Hall's Theorem

No bottlenecks implies
there is a perfect match.

proof:
by strong induction
on # girls

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Proof of Hall's Theorem

Case 1: there is a nonempty
proper subset S of girls with
 $|S| = |E(S)|$.

by Lemmas, no bottlenecks in
Hall graph $(S, E(S))$,
and none in $(\overline{S}, \overline{E(S)})$

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6	9	13	7
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Proof of Hall's Theorem

by induction, match
 $(S, E(S))$ and $(\overline{S}, \overline{E(S)})$
separately.

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Proof of Hall's Theorem

by induction, match $(S, E(S))$ and $(\overline{S}, E(\overline{S}))$ separately. Matchings don't overlap, so union is a complete matching.



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Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S . Pick a girl, g .



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Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S . Pick a girl, g . She must be compatible with some boy, b (in fact, at least 2 boys).



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Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S . Match g with b .



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Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S .
 Match g with b . Removing b still leaves $|S| \leq |E(S)|$, so no bottlenecks.



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Hall.21

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Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S .
 By induction, can match remaining girls & boys. This match along with $g-b$ is complete match. QED



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Hall.22

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