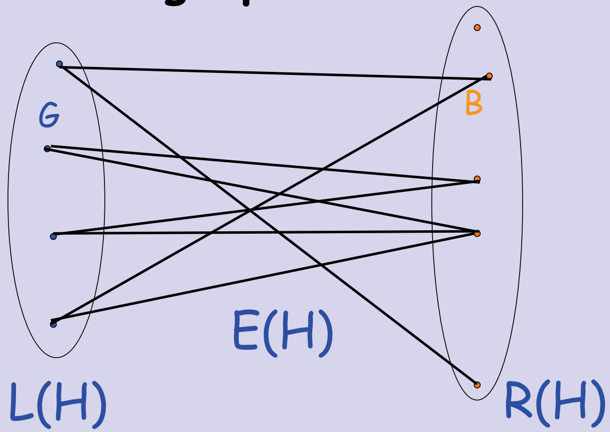
Mathematics for Computer Science

MIT 6.042J/18.062J

Hall's Theorem

Hall graph H



Hall graph H

A match is a

total injective function

$m:G \to B$

that follows edges

Hall graph H

A match is a

total injective function

$m:G \to B$

$g - m(g) \in E$

Hall graph H

A match is a total injective function

$$m:G\to B$$

$$\text{graph}(m) \subseteq E$$

Hall's Theorem

Hall's condition

If $|S| \leq |E(S)|$ for all sets of girls, S, then there is a match.

How to verify no bottlenecks?

fairly efficient matching procedure is known

(explained in algorithms subjects)

...but there is a special situation that ensures a match...

How to verify no bottlenecks?

If every girl likes $\geq$ d boys, and every boy likes $\leq$ d girls, then no bottlenecks.

a degree-constrained Hall graph

How to verify no bottlenecks?

If every girl likes $\geq d$ boys,
and every boy likes $\leq d$ girls,
then no bottlenecks.
proof:

How to verify no bottlenecks?

If every girl likes $\geq d$ boys,
and every boy likes $\leq d$ girls,
then no bottlenecks.
proof: say set S of girls has e incident edges:

$$d \cdot |S| \leq e \leq d \cdot |E(S)|$$

$$\text{so } |S| \leq |E(S)|$$

no bottleneck

Proof of Hall's Theorem

Suppose no bottlenecks.

Lemma: No bottlenecks within any set S of girls.

obviously

Proof of Hall's Theorem

Suppose no bottlenecks.

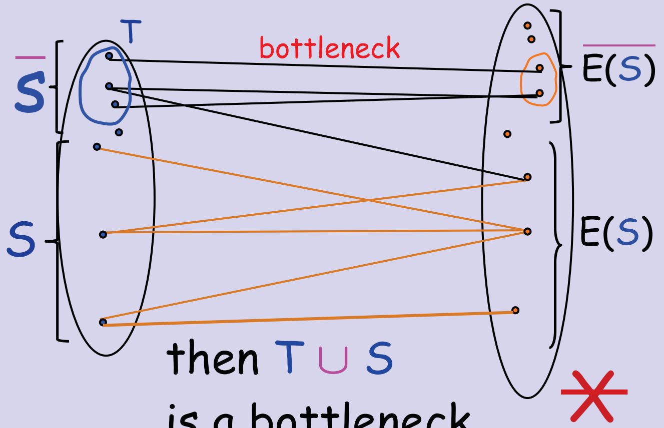
Lemma: If S a set of girls with

$$|S| = |E(S)|,$$

then no bottlenecks between

$\overline{S}$ and $\overline{E(S)}$ either

bottleneck between $\overline{S}$ & $\overline{E(S)}$?



then T ∪ S

is a bottleneck

Proof of Hall's Theorem

No bottlenecks implies there is a perfect match.

proof:

by strong induction on # girls

Proof of Hall's Theorem

Case 1: there is a nonempty proper subset S of girls with

$$|S|=|E(S)|.$$

by Lemmas, no bottlenecks in Hall graph (S, E(S)), and none in $(\overline{S}, \overline{E(S)})$

Proof of Hall's Theorem

by induction, match (S, E(S)) and $(\overline{S}, \overline{E(S)})$ separately.

Proof of Hall's Theorem

by induction, match $(S, E(S))$ and $(\overline{S}, \overline{E(S)})$ separately. Matchings don't overlap, so union is a complete matching.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.
Pick a girl, g.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.
Pick a girl, g. She must be compatible with some boy, b (in fact, at least 2 boys).

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.
Match g with b.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.

Match g with b. Removing b still leaves $|S| \leq |E(S)|$, so no bottlenecks.

Hall's Theorem

Case 2: $|S| < |E(S)|$ for all nonempty proper subsets S.

By induction, can match remaining girls & boys. This match along with g—b is complete match. QED