

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Variance



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Variance of an Indicator

I an indicator with $E[I]=p$:

$$\begin{aligned} \text{Var}[I] &::= E[(I - p)^2] \\ &= E[I^2] - 2pE[I] + p^2 \\ &= E[I] - 2p \cdot p + p^2 \\ &= p - 2p^2 + p^2 = pq \end{aligned}$$



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Calculating Variance

$$\begin{aligned} \text{Var}[aR + b] &= a^2 \text{Var}[R] \\ \text{Var}[R] &= E[R^2] - (E[R])^2 \end{aligned}$$



6	9	13	7
12	10	5	
3	1	4	14
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Calculating Variance

$$\begin{aligned} \text{Var}[aR + b] &= a^2 \text{Var}[R] \\ \text{Var}[R] &= E[R^2] - E^2[R] \end{aligned}$$

simple proofs applying linearity
of $E[\]$ to the def of $\text{Var}[\]$





proof of 2nd Variance Formula

$$\begin{aligned} \text{Var}[R] &::= E[(R - \mu)^2] \\ &= E[R^2 - 2\mu R + \mu^2] \\ &= E[R^2] - 2\mu \cdot E[R] + E[\mu^2] \\ &= E[R^2] - 2\mu \cdot \mu + \mu^2 \\ &= E[R^2] - \mu^2 \\ &= E[R^2] - E^2[R] \end{aligned}$$


Albert R Meyer, May 10, 2013 variance.5



Space Station Mir

Destructs with probability p
in any given hour

$E[F] = 1/p$ (Mean Time to Fail)

$\text{Var}[F] = ?$



Albert R Meyer, May 10, 2013 variance.6



Variance of Time to Failure

$$\begin{aligned} \Pr[F = k] &= q^{k-1}p \\ \text{Var}[F] &= E[F^2] - E^2[F] \\ F &= 1, 2, 3, \dots, k, \dots \\ F^2 &= 1, 4, 9, \dots, k^2, \dots \end{aligned}$$


Albert R Meyer, May 10, 2013 variance.7



Variance of Time to Failure

$$\begin{aligned} E[F^2] &::= \sum_{k=1}^{\infty} k^2 \cdot \Pr[F^2 = k^2] \\ &= \sum_{k=1}^{\infty} k^2 \cdot \Pr[F = k] \\ &= \frac{p}{q} \sum_{k=0}^{\infty} k^2 q^k \end{aligned}$$

has closed form



Albert R Meyer, May 10, 2013 variance.8



Variance of Time to Failure
total expectation $E[F^2]=$
approach:

$$E[F^2 | F = 1] \cdot \Pr[F = 1] \\ + E[F^2 | F > 1] \cdot \Pr[F > 1]$$


Albert R Meyer, May 10, 2013 variance.9



Conditional time to failure
lemma: For $F =$ time to failure, $g: \mathbb{R} \rightarrow \mathbb{R}$,

$$E[g(F) | F > n]$$


Albert R Meyer, May 10, 2013 variance.10



Conditional time to failure
lemma: For $F =$ time to failure, $g: \mathbb{R} \rightarrow \mathbb{R}$,

$$E[g(F) | F > n] = E[g(F + n)]$$

Corollary:

$$E[F^2 | F > 1] = E[(F + 1)^2]$$


Albert R Meyer, May 10, 2013 variance.11



Variance of Time to Failure
total expectation $E[F^2]=$
approach:

$$E[F^2 | F = 1] \cdot \Pr[F = 1] \\ + E[F^2 | F > 1] \cdot \Pr[F > 1]$$


Albert R Meyer, May 10, 2013 variance.12


 Variance of Time to Failure
 total expectation approach: $E[F^2] =$
 $1 \cdot p$
 $+ E[F^2 | F > 1] \cdot \Pr[F > 1]$


 Albert R Meyer, May 10, 2013 variance.13


 Variance of Time to Failure
 total expectation approach: $E[F^2] =$
 $1 \cdot p$
 $+ E[(F + 1)^2] \cdot q$


 Albert R Meyer, May 10, 2013 variance.14


 Variance of Time to Failure
 total expectation approach: $E[F^2] =$
 $1 \cdot p$
 $+ (E[F^2] + 2/p + 1) \cdot q$
 now solve for $E[F^2]$


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 Mean Time to Failure

$$\text{Var}[F] = \frac{1}{p} \left(\frac{1}{p} - 1 \right)$$

Mir1:
 $p = 10^{-4}, E[F] = 10^4, \sigma < 10^4$
 so by Chebyshev
 $\Pr[\text{lasts} \geq 4 \cdot 10^4 \text{ hours}] \leq 1/4$


 Albert R Meyer, May 10, 2013 variance.18



Mean Time to Failure

$$\text{Var}[F] = \frac{1}{p} \left(\frac{1}{p} - 1 \right)$$

Mir1:

$p = 10^{-4}, E[F] = 10^4, \sigma < 10^4$
 so by Chebyshev

$\text{Pr}[\text{lasts} \geq 4.6 \text{ years}] \leq 1/4$



Albert R Meyer, May 10, 2013 variance.19



Calculating Variance

Pairwise Independent Additivity

$$\begin{aligned} &\text{Var}[R_1 + R_2 + \dots + R_n] \\ &= \text{Var}[R_1] + \text{Var}[R_2] + \dots + \text{Var}[R_n] \end{aligned}$$

providing R_1, R_2, \dots, R_n are
 pairwise independent

again, a simple proof applying
 linearity of $E[\]$ to the def of $\text{Var}[\]$



Albert R Meyer, May 10, 2013 variance.20

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