Mathematics for Computer Science

MIT 6.042J/18.062J

Variance

Variance of an Indicator

I an indicator with $E[I]=p$:

$$\mathrm{Var}[I] ::= E[(I-p)^2]$$
$$= E[I^2] - 2pE[I] + p^2$$
$$= E[I] - 2p \cdot p + p^2$$
$$= p - 2p^2 + p^2 = pq$$

Calculating Variance

$$\mathrm{Var}[aR+b] = a^2\,\mathrm{Var}[R]$$
$$\mathrm{Var}[R] = E[R^2] - (E[R])^2$$

Calculating Variance

$$\mathrm{Var}[aR+b] = a^2\,\mathrm{Var}[R]$$
$$\mathrm{Var}[R] = E[R^2] - E^2[R]$$

simple proofs applying linearity of E[] to the def of Var[]

proof of 2nd Variance Formula

$Var[R] ::= E[(R - \mu)^2]$

$= E[R^2 - 2\mu R + \mu^2]$

$= E[R^2] - 2\mu \cdot E[R] + E[\mu^2]$

$= E[R^2] - 2\mu \cdot \mu + \mu^2$

$= E[R^2] - \mu^2$

$= E[R^2] - E^2[R]$

Space Station Mir

Destructs with probability p in any given hour

$E[F] = 1/p$ (Mean Time to Fail)

$Var[F] = ?$

Variance of Time to Failure

$Pr[F = k] = q^{k-1}p$

$Var[F] = E[F^2] - E^2[F]$

$F = 1, 2, 3, \ldots, k, \ldots$

$F^2 = 1, 4, 9, \ldots, k^2, \ldots$

Variance of Time to Failure

$$E[F^2] ::= \sum_{k=1}^{\infty} k^2 \cdot Pr[F^2 = k^2]$$

$$= \sum_{k=1}^{\infty} k^2 \cdot Pr[F = k]$$

$$= \frac{p}{q} \underbrace{\sum_{k=0}^{\infty} k^2 q^k}_{\text{has closed form}}$$

has closed form

Variance of Time to Failure

total expectation approach: $E[F^2] =$

$$E[F^2 \mid F=1]\cdot \Pr[F=1]$$

$$+ E[F^2 \mid F>1]\cdot \Pr[F>1]$$

Conditional time to failure

lemma: For F = time to failure, $g:\mathbb{R}\to\mathbb{R}$,

$$E[\,g(F) \mid F>n]$$

Conditional time to failure

lemma: For F = time to failure, $g:\mathbb{R}\to\mathbb{R}$,

$$E[\,g(F) \mid F>n] = E[\,g(F+n)\,]$$

Corollary:

$$E[F^2 \mid F>1] = E[(F+1)^2]$$

Variance of Time to Failure

total expectation approach: $E[F^2] =$

$$E[F^2 \mid F=1]\cdot \Pr[F=1]$$

$$+ E[F^2 \mid F>1]\cdot \Pr[F>1]$$

Variance of Time to Failure

total expectation approach:

$E[F^2] =$

$1 \cdot p$

$+ E[F^2 \mid F > 1] \cdot Pr[F > 1]$

Variance of Time to Failure

total expectation approach:

$E[F^2] =$

$1 \cdot p$

$+ E[(F+1)^2] \cdot q$

Variance of Time to Failure

total expectation approach:

$E[F^2] =$

$1 \cdot p$

$+ (E[F^2] + 2/p + 1)$

$\cdot q$

now solve for $E[F^2]$

Mean Time to Failure

$$Var[F] = \frac{1}{p}\left(\frac{1}{p} - 1\right)$$

Mir1: $p = 10^{-4}$, $E[F] = 10^4$, $\sigma < 10^4$

so by Chebyshev

Pr[lasts $\geq 4\ 10^4$ hours] $\leq 1/4$

Mean Time to Failure

$$\mathrm{Var}[F] = \frac{1}{p}\left(\frac{1}{p} - 1\right)$$

Mir1:

$p = 10^{-4}$, $E[F] = 10^{4}$, $\sigma < 10^{4}$

so by Chebyshev

Pr[lasts $\geq$ 4.6 years] $\leq$ 1/4

Calculating Variance

Pairwise Independent Additivity

$$\mathrm{Var}[R_1 + R_2 + \cdots + R_n]$$
$$= \mathrm{Var}[R_1] + \mathrm{Var}[R_2] + \cdots + \mathrm{Var}[R_n]$$

providing $R_1, R_2, \ldots, R_n$ are pairwise independent

again, a simple proof applying linearity of E[] to the def of Var[]