Mathematics for Computer Science
MIT 6.042J/18.062J

Random Variables Independence

Random Variables

Informally: an RV is a number produced by a random process:

- # hours to next system crash
- # faulty pixels in monitor
- # alpha particles in a second
- # heads in n coin flips

Intro to Random Variables

Example: Flip three fair coins

C ::= # heads (Count)

$$M ::= \begin{cases} 1 & \text{if all Match,} \\ 0 & \text{otherwise.} \end{cases}$$

Intro to Random Variables

Specify events using values of variables

- [C = 1] is event "exactly 1 head"
  Pr[C = 1] = 3/8
- Pr[$C \geq 1$] = 7/8
- Pr[$C \cdot M > 0$] = Pr[M>0 AND C>0]
  = Pr[all heads] = 1/8

What is a Random Variable?

Formally,

$$R: \mathcal{S} \to \mathbb{R}$$

Sample space

(usually)

What is a Random Variable?

R packages together the events $[R = a]$ for $a \in \mathbb{R}$

Event properties carry over to RV's directly

Mutally Independent Variables

Def: $R_1, R_2, \ldots, R_n$

are mutually indep RV's iff

$[R_1=a_1],[R_2=a_2],\ldots,[R_n=a_n]$

are mutually indep events

for all $a_1, a_2, \ldots, a_n$

Mutally Independent Variables

Alternatively:

$$\Pr[R_1=a_1 \text{ AND } R_2=a_2 \text{ AND } \cdots \text{ AND } R_n=a_n]$$

$$= \Pr[R_1=a_1] \cdot \Pr[R_2=a_2] \cdot \cdots \Pr[R_n=a_n]$$

Independent Variables

Are C and M

independent? NO

$\Pr[M=1] \cdot \Pr[C=1] > 0$

$\Pr[M=1 \text{ and } C=1] = 0$

Indicator Variables

The indicator variable for event A:

$$I_A ::= \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

(Sanity check:

I_A and I_B are independent iff A and B are independent)

Independent Variables

O ::= odd #Heads

Are M and I_O

independent? YES

(Work it out!)

Independent Variables

Lemma:

If R is independent of S, then R is independent of any information about S.

Independent Variables

Lemma:

If R is independent of S, and $f:\mathbb{R}\to\mathbb{R}$, then R is independent of f(S)

k-way Independent Variables

k-way Independence: any k of the variables are mutually independent

2-way is called pairwise

k-way Independent Variables

H_i ::= indicator for Head on flip $i \in [1,k]$

$O ::= \oplus_{i=1}^{k} H_i$ (mod 2 sum).

Any k of them are independent, but not k+1-way independent since any k determine the remaining one.

Pairwise Independent Variables

Pairwise Independence sufficient for major applications (in later lecture).

Good to know, since pairwise holds in important cases where mutual does not.