



Mathematics for Computer Science  
MIT 6.042J/18.062J

# Random Variables Independence

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# Random Variables

Informally: an **RV** is a number produced by a **random process**:

- # hours to next system crash
- # faulty pixels in monitor
- # alpha particles in a second
- # heads in **n** coin flips

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# Intro to Random Variables

Example: Flip three fair coins

**C** ::= # heads (**C**ount)

**M** ::=  $\begin{cases} 1 & \text{if all Match,} \\ 0 & \text{otherwise.} \end{cases}$

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# Intro to Random Variables

Specify events using values of variables

- $[C = 1]$  is event "exactly 1 head"  
 $\Pr[C = 1] = 3/8$
- $\Pr[C \geq 1] = 7/8$
- $\Pr[C \cdot M > 0] = \Pr[M > 0 \text{ AND } C > 0]$   
 $= \Pr[\text{all heads}] = 1/8$

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What is a Random Variable?

Formally,

$$R: \mathcal{S} \rightarrow \mathbb{R}$$

Sample space (usually)

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What is a Random Variable?

$\mathbb{R}$  packages together the events  $[R = a]$  for  $a \in \mathbb{R}$   
 Event properties carry over to RV's directly

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Mutally Independent Variables

Def:  $R_1, R_2, \dots, R_n$   
 are mutually indep RV's iff  
 $[R_1=a_1], [R_2=a_2], \dots, [R_n=a_n]$   
 are mutually indep events  
 for all  $a_1, a_2, \dots, a_n$

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Mutally Independent Variables  
 Alternatively:

$$\Pr[R_1=a_1 \text{ AND } R_2=a_2 \text{ AND } \dots \text{ AND } R_n=a_n] \\ = \Pr[R_1=a_1] \cdot \Pr[R_2=a_2] \cdot \dots \Pr[R_n=a_n]$$

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### Independent Variables

Are  $C$  and  $M$  independent? **NO**

$$\Pr[M=1] \cdot \Pr[C=1] > 0$$

$$\Pr[M=1 \text{ and } C=1] = 0$$


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### Indicator Variables

The indicator variable for event  $A$ :

$$I_A ::= \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

(Sanity check:  
 $I_A$  and  $I_B$  are independent iff  
 $A$  and  $B$  are independent)



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### Independent Variables

$O ::= \text{odd \#Heads}$

Are  $M$  and  $I_O$  independent? **YES**

(Work it out!)



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### Independent Variables

**Lemma:**  
 If  $R$  is independent of  $S$ ,  
 then  $R$  is independent of  
 any information about  $S$ .



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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Independent Variables

Lemma:

If  $R$  is independent of  $S$ ,  
and  $f: \mathbb{R} \rightarrow \mathbb{R}$ , then  
 $R$  is independent of  $f(S)$



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6	9	13	7
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## $k$ -way Independent Variables

$k$ -way Independence:  
any  $k$  of the variables are  
mutually independent  
2-way is called pairwise



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6	9	13	7
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## $k$ -way Independent Variables

$H_i ::=$  indicator for Head on flip  $i \in [1, k]$

$O ::= \bigoplus_{i=1}^k H_i \pmod{2}$  (mod 2 sum).

Any  $k$  of them are independent,  
but **not**  $k+1$ -way independent  
since any  $k$  determine the  
remaining one.



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6	9	13	7
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15	8	11	2

## Pairwise Independent Variables

Pairwise Independence sufficient  
for major applications (in later  
lecture).

Good to know, since pairwise holds  
in important cases where mutual  
does not.



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6.042J / 18.062J Mathematics for Computer Science  
Spring 2015

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