

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science  
 MIT 6.042J/18.062J

# Partial Orders

Albert R Meyer March 22, 2013

po's.1

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## Walks in digraph $G$

walk from  $u$  to  $v$  and  
 from  $v$  to  $w$



implies walk from  $u$  to  $w$

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po's.2

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## Walks in digraph $G$

walk from  $u$  to  $v$  and  
 from  $v$  to  $w$ , implies  
 walk from  $u$  to  $w$ :

$u \in G^+ v$  AND  $v \in G^+ w$   
 IMPLIES  $u \in G^+ w$

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po's.3

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## Walks in digraph $G$

transitive relation  $R$ :

$u R v$  AND  $v R w$   
 IMPLIES  $u R w$

$G^+$  is transitive

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po's.4

6	9	13	7
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## transitivity

*Theorem:*  
 $R$  is a transitive iff  
 $R = G^+$  for some digraph  $G$

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## Paths in DAG $D$

pos length path from  $u$  to  $v$  implies  
 no path from  $v$  to  $u$   
 $u D^+ v$  IMPLIES NOT( $v D^+ u$ )

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## Paths in DAG $D$

asymmetric relation  $R$ :  
 $u R v$  IMPLIES NOT( $v R u$ )  
 $D^+$  is asymmetric

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## strict partial orders

# transitive & asymmetric

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## strict partial orders

examples:

- $\subset$  on sets
- "indirect prerequisite" on MIT subjects
- less than,  $<$ , on real numbers



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po's.9

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## strict partial orders

Theorem:

$R$  is a SPO iff

$R = D^+$  for some  
DAG  $D$



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po's.10

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## linear orders

Given any two elements,  
one will be "bigger than"  
the other one.



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po's.11

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## linear orders

basic example:

$<$  or  $\leq$  on the Reals:  
if  $x \neq y$ , then either

$x < y$  OR  $y < x$



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po's.12

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## linear orders

$\mathbb{R}$  is linear:  
 no incomparable elements

if  $x \neq y$ , then either  
 $x R y$  OR  $y R x$

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## linear orders

The whole partial order is a chain



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## linear orders

A topological sort turns  
 a partial order into a  
 linear order ...in a way  
 that is consistent  
 with the partial order

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## weak partial orders

same as a strict partial  
 order  $R$ , except that  
 $a R a$  always holds

examples:

- $\leq$  is weak p.o. on  $\mathbb{R}$
- $\subseteq$  is weak p.o. on sets

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## reflexivity

relation  $R$  on set  $A$   
 is **reflexive** iff  
 $aRa$  for all  $a \in A$   
 $G^*$  is reflexive

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## antisymmetry

binary relation  $R$  is  
**antisymmetric** iff  
 it is asymmetric  
 except for  $aRa$  case.

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## A/Antisymmetry

minor difference:  
 whether  $aRa$  is allowed

never                      sometimes

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## antisymmetry

**antisymmetric** relation  $R$ :

$uRv$  IMPLIES **NOT**  $(vRu)$   
 for  $u \neq v$

$D^*$  is antisymmetric for  
 DAG  $D$

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weak partial orders

transitive,  
antisymmetric &  
reflexive



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po's.21

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weak partial orders

Theorem:

$R$  is a WPO iff

$R = D^*$  for some  
DAG  $D$



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po's.22

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