


**Mathematics for Computer Science**  
 MIT 6.042J/18.062J

# Independent Sampling Theorem


 Albert R Meyer, May 13, 2013 sampletheorem.1


**Weak Law of Large Numbers**

$A_n ::=$  avg of  $n$  indep RV's with mean  $\mu$

**Theorem:** For all  $\delta > 0$

$$\lim_{n \rightarrow \infty} \Pr[ |A_n - \mu| > \delta ] = 0$$

**Proof:**


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**Weak Law of Large Numbers**

$A_n ::=$  avg of  $n$  indep RV's with mean  $\mu$ , var  $\sigma^2$

**Theorem:** For all  $\delta > 0$

$$\lim_{n \rightarrow \infty} \Pr[ |A_n - \mu| > \delta ] = 0$$

**Proof:**


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**Repeated Trials**

$$\begin{aligned}
 E[A_n] &::= E \left[ \frac{R_1 + R_2 + \dots + R_n}{n} \right] \\
 &= \frac{E[R_1] + E[R_2] + \dots + E[R_n]}{n} \\
 &= \frac{n\mu}{n} = \mu
 \end{aligned}$$


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 **Weak Law of Large Numbers**

So by Chebyshev

$$\Pr[|A_n - \mu| > \delta] \leq \frac{\text{Var}[A_n]}{\delta^2}$$

need only show

$$\text{Var}[A_n] \rightarrow 0 \text{ as } n \rightarrow \infty$$

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 **Var[ $A_n$ ]**

$$\text{Var}[A_n] = \text{Var}\left[\frac{R_1 + R_2 + \dots + R_n}{n}\right]$$

$$= \frac{\text{Var}[R_1] + \text{Var}[R_2] + \dots + \text{Var}[R_n]}{n^2}$$

**QED**  $= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \rightarrow 0$

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 **Analysis of the Proof**

proof only used that  $R_1, \dots, R_n$  have

- same mean
- same variance
- & variances add

— which follows from **pairwise** independence

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 **Pairwise Independent Sampling**

**Theorem:**  
Let  $R_1, \dots, R_n$  be **pairwise independent** random vars with the same finite mean  $\mu$  and variance  $\sigma^2$ . Let  $A_n ::= (R_1 + R_2 + \dots + R_n)/n$ . Then

$$\Pr[|A_n - \mu| > \delta] \leq \frac{1}{n} \left(\frac{\sigma}{\delta}\right)^2$$

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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Pairwise Independent Sampling

The punchline:

we now know how big a sample is needed to estimate the mean of any\* random variable within any\* desired tolerance with any\* desired probability

\*variance  $< \infty$ , tolerance  $> 0$ , probability  $< 1$



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