



Mathematics for Computer Science  
MIT 6.042J/18.062J

# Great Expectations



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## Carnival Dice

Choose a number from 1 to 6,  
then roll 3 fair dice:

win \$1 for each match  
lose \$1 if no match



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## Carnival Dice

*Example:* choose 5, then

roll 2,3,4: lose \$1  
roll 5,4,6: win \$1  
roll 5,4,5: win \$2  
roll 5,5,5: win \$3



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## Carnival Dice

# Is this a fair game?



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## Carnival Dice

$$\Pr[0 \text{ fives}] = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$\Pr[1 \text{ five}] = \binom{3}{1} \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)$$

$$\Pr[2 \text{ fives}] = \binom{3}{2} \left(\frac{5}{6}\right)^1 \left(\frac{1}{6}\right)^2$$

$$\Pr[3 \text{ fives}] = \left(\frac{1}{6}\right)^3$$

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## Carnival Dice

# matches	probability	\$ won
0	125/216	-1
1	75/216	1
2	15/216	2
3	1/216	3

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## Carnival Dice

so every 216 games, expect

- 0 matches about 125 times
- 1 match about 75 times
- 2 matches about 15 times
- 3 matches about once

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## Carnival Dice

So on average expect to win:

$$\frac{125 \cdot (-1) + 75 \cdot 1 + 15 \cdot 2 + 1 \cdot 3}{216}$$

$$= -\frac{17}{216} \approx -8 \text{ cents}$$

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## Carnival Dice

So on average expect to win:

$$\frac{125 \cdot \boxed{\text{NOT fair!}} \cdot 1.3}{216} = -\frac{17}{216} \approx -8\text{cents}$$

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## Carnival Dice

You can "expect" to lose **8 cents** per play. But you **never actually** lose **8 cents** on any single play, this is just your **average loss**.

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## Expected Value

The **expected value** of random variable **R** is the **average value** of **R** --with values weighted by their probabilities

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## Expected Value

The **expected value** of random variable **R** is

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$

so  $E[\text{\$win in Carnival}] = -\frac{17}{216}$

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**Alternative definition**

$$E[R] = \sum_{\omega \in \mathcal{S}} R(\omega) \cdot \Pr[\omega]$$

this form helpful in some proofs



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**Alternative definition**

$$E[R] = \sum_{\omega \in \mathcal{S}} R(\omega) \cdot \Pr[\omega]$$

proof of equivalence:  
 $[R = v] ::= \{\omega \mid R(\omega) = v\}$  so

$$\Pr[R = v] ::= \sum_{\omega \in [R=v]} \Pr[\omega]$$


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**proof of equivalence**

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$


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**proof of equivalence**

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$


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**proof of equivalence**

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega]$$


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**proof of equivalence**

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega]$$

$$= \sum_v \sum_{\omega \in [R=v]} v \cdot \text{Pr}[\omega]$$


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**proof of equivalence**

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega]$$

$$= \sum_v \sum_{\omega \in [R=v]} v \cdot \text{Pr}[\omega]$$


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**proof of equivalence**

Now

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \sum_{\omega \in [R=v]} \text{Pr}[\omega]$$

$$= \sum_v \sum_{\omega \in [R=v]} R(\omega) \cdot \text{Pr}[\omega]$$

$$= \sum_{\omega \in S} R(\omega) \cdot \text{Pr}[\omega]$$


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## Sums vs Integrals

We get away with sums instead of integrals because the sample space is assumed **countable**:

$$S = \{\omega_0, \omega_1, \dots, \omega_n, \dots\}$$


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## Rearranging Terms

It's safe to rearrange terms in sums because



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## Rearranging Terms

It's safe to rearrange terms in sums because we implicitly assume that the defining sum for the expectation is **absolutely convergent**



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## Absolute convergence

$$E[R] ::= \sum_{v \in \text{range}(R)} v \cdot \Pr[R = v]$$

the terms on the right could be rearranged to equal anything at all when the sum is **not** absolutely convergent



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## Expected Value

also called

mean value, mean, or  
expectation



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## Expectations & Averages

From a pile of graded exams,  
pick one at random, and let  $S$   
be its score.



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## Expectations & Averages

From a pile of graded exams,  
pick one at random, and let  $S$   
be its score. Now  $E[S]$  equals  
the average exam score



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## Expectations & Averages

We can estimate averages  
by estimating expectations  
of random variables



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## Expectations & Averages

We can estimate **averages**  
by estimating **expectations**  
of random variables based  
on picking random elements

**sampling**



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## Expectations & Averages

For example, it is impossible for  
all exams to be above average  
(no matter what the townspeople  
of Lake Woebegone say):

$$\Pr[R > E[R]] < 1$$



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## Expectations & Averages

On the other hand

$$\Pr[R > E[R]] \geq 1 - \epsilon$$

is possible for all  $\epsilon > 0$

For example, almost  
everyone has an above  
average number of fingers.



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