


**Mathematics for Computer Science**  
 MIT 6.042J/18.062J

# Conditional Probability


 Albert R Meyer, May 3, 2013 condprob.1


**Conditional Probability: A Fair Die**

$$\Pr[\text{roll } 1] = \frac{|\{1\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{6}$$

“knowledge” changes probabilities:  
 $\Pr[\text{roll } 1 \text{ knowing rolled odd}]$

$$= \frac{|\{1\}|}{|\{1,3,5\}|} = \frac{1}{3}$$


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**Conditional Probability: A Fair Die**

$\Pr[\text{one} \mid \text{odd}] = \frac{1}{3}$  (Yes)  $\Pr: \frac{1}{6}$   
 $\Pr[\text{not one} \mid \text{odd}] = \frac{2}{3}$  (No)  $\frac{1}{3}$   
 $\Pr[\text{not one} \mid \text{even}] = \frac{1}{1}$  (No)  $\frac{1}{2}$

Rolled odd      Rolled 1


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**Monty Hall Probabilities**

$\Pr[\text{pick } 1 \mid \text{prize } 1] = \frac{1}{3}$   
 $\Pr[\text{open } 3 \mid \text{prize } 1 \text{ \& pick } 1] = \frac{1}{2}$   
 $\Pr[\text{pick } 2 \mid \text{prize } 3] = \frac{1}{3}$


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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Conditional Probability

We were reasoning about conditional probability in the way we defined our probability spaces in the first place.

We were using:



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6	9	13	7
12	10	5	
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15	8	11	2

## Product Rule

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B | A]$$



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condprob.8

6	9	13	7
12	10	5	
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15	8	11	2

## Conditional Probability

In fact, we use this reasoning to **define** conditional probability:



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6	9	13	7
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## Conditional Probability

$\Pr[B|A]$  is the probability of event  $B$ , **given** that event  $A$  has occurred:

$$\Pr[B | A] ::= \frac{\Pr[A \cap B]}{\Pr[A]}$$



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6	9	13	7
12		10	5
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15	8	11	2

## Product Rule for 3

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B | A] \cdot \Pr[C | A \cap B]$$



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Conditioning Defines a New Space

Conditioning on  $A$  defines a new probability function  $\Pr_A$  where



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condprob.12

6	9	13	7
12		10	5
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## Conditioning Defines a New Space

Conditioning on  $A$  defines a new probability function  $\Pr_A$  where outcomes not in  $A$  are assigned probability **zero**



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Conditioning Defines a New Space

Conditioning on  $A$  defines a new probability function  $\Pr_A$  where outcomes not in  $A$  are assigned probability **zero**, and outcomes in  $A$  have their probabilities raised in proportion to  $A$ .



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condprob.14

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Conditioning Defines a New Space

Conditioning on  $A$  defines a new probability function  $\Pr_A$  where

$$\Pr_A[\omega] ::= 0 \quad \text{if } \omega \notin A,$$

$$::= \frac{\Pr[\omega]}{\Pr[A]} \quad \text{if } \omega \in A.$$



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Conditioning Defines a New Space

Now

$$\Pr[B | A] = \Pr_A[B].$$

This implies conditional probability obeys all the rules, for example

**Conditional Difference Rule**

$$\Pr[B - C | A] =$$

$$\Pr[B | A] - \Pr[B \cap C | A]$$



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