Mathematics for Computer Science
MIT 6.042J/18.062J

Conditional Probability

Conditional Probability: A Fair Die

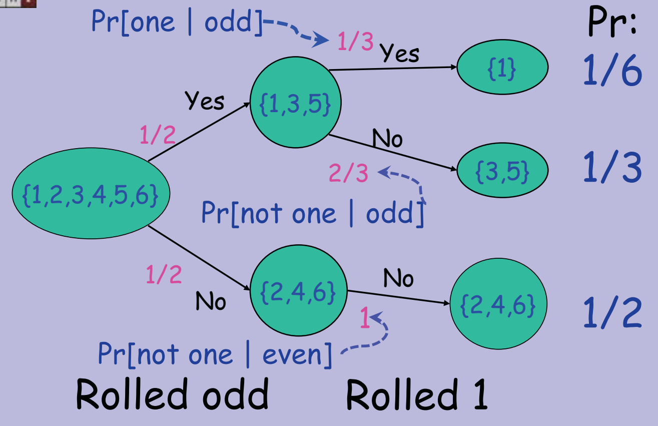
$$\Pr[\text{roll } 1] = \frac{|\{1\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{6}$$

"knowledge" changes probabilities:

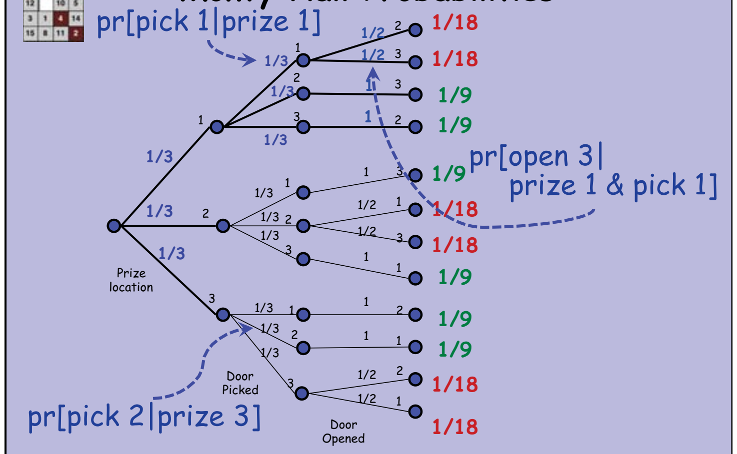
Pr[roll 1 knowing rolled odd]

$$= \frac{|\{1\}|}{|\{1,3,5\}|} = \frac{1}{3}$$

Conditional Probability: A Fair Die



Monty Hall Probabilities



Conditional Probability

We were reasoning about conditional probability in the way we defined our probability spaces in the first place.

We were using:

Product Rule

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$$

Conditional Probability

In fact, we use this reasoning to *define* conditional probability:

Conditional Probability

$\Pr[B|A]$ is the probability of event B, *given* that event A has occurred:

$$\Pr[B \mid A] ::= \frac{\Pr[A \cap B]}{\Pr[A]}$$

Product Rule for 3

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B \mid A] \cdot \Pr[C \mid A \cap B]$$

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Conditioning on A defines a new probability function $\Pr_A$ where

$$\Pr_A[\omega] ::= 0 \qquad \text{if } \omega \notin A,$$

$$::= \frac{\Pr[\omega]}{\Pr[A]} \qquad \text{if } \omega \in A.$$

Conditioning Defines a New Space

Now

$$\Pr[B \mid A] = \Pr_A[B].$$

This implies conditional probability obeys all the rules, for example

Conditional Difference Rule

$$\Pr[B - C \mid A] = \Pr[B \mid A] - \Pr[B \cap C \mid A]$$