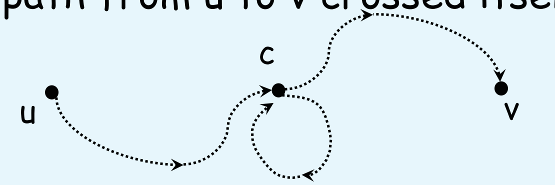
Mathematics for Computer Science
MIT 6.042J/18.062J

Connected vertices

Walks & Paths

Lemma:
The shortest walk between two vertices is a path!

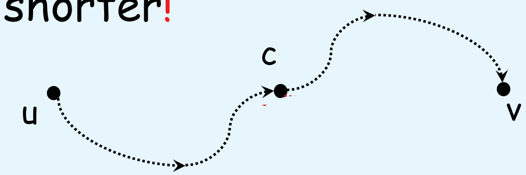
Proof: (by contradiction) suppose path from u to v crossed itself:



Walks & Paths

Lemma:
The shortest walk between two vertices is a path!

then path without c---c is shorter!



length n walk relation

$v \, G^n \, w$

IFF ∃ length n walk from v to w

G^n is the length n walk relation for G

length n walk relation

G itself is the length 1 walk relation: $G^1 = G$

lemma:

$$\underbrace{G^m \circ G^n}_{\text{relational composition}} = G^{m+n}$$

$G^m \circ G^n = G^{m+n}$

$x(G^m \circ G^n)y$ ::= $\exists z.\ x\, G^m\, z\, G^n\, y$

IFF $x\, G^{m+n}\, y$

because a length m+n walk consists of a length m walk followed by a length n walk

Matrices & Composition

A_G ::= Adjacency matrix for G

Lemma: $A_{G \circ H} = A_H \odot A_G$

where $\odot$ is Boolean matrix product—using OR instead of +

Matrices & Composition

So compute A_{G^n} by fast *matrix* exponentiation

$\approx$ log n matrix products.

Walk Relation

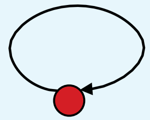
G^* is walk relation of G

$u \; G^* \; v$ iff $\exists$walk u to v

(u is connected to v)

Compute the Walk Relation

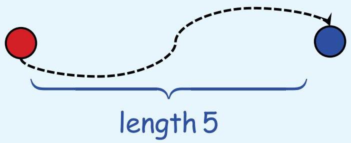
Add self-loops:



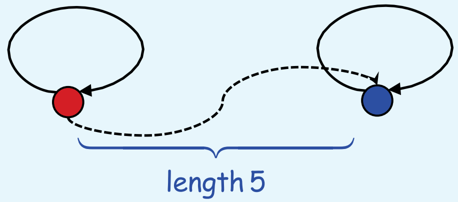
$$G^{\leq} ::= G \cup Id_V$$

$G^{\leq}$ has a length n walk iff G has a length $\leq n$ walk

lengthening a walk in G



lengthening a walk in $G^{\leq}$



just keep looping k times to make a length $5+k$ walk in $G^{\leq}$

Compute the Walk Relation

If G has n vertices, then

length of paths is $< n$, and

$$G^* = \left(G^{\leq}\right)^{n-1}$$

So find all connected vertex pairs with $n^2 \log n$ AND/OR's

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