

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Mathematics for Computer Science
MIT 6.042J/18.062J

Connected vertices



Albert R Meyer March 15, 2013

connected.1

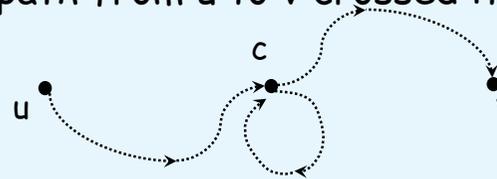
| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Walks & Paths

Lemma:

The **shortest** walk between
two vertices is a path!

Proof: (by contradiction) suppose
path from u to v crossed itself:



Albert R Meyer March 15, 2013

connected.2

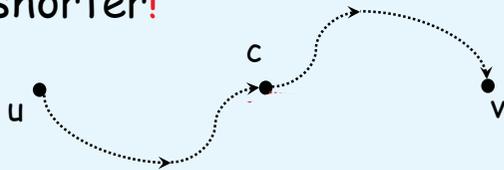
| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Walks & Paths

Lemma:

The **shortest** walk between
two vertices is a path!

then path without $c \text{---} c$ is
shorter!



Albert R Meyer March 15, 2013

connected.3

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

length n walk relation

$$v G^n w$$

IFF \exists length n walk
from v to w

G^n is the length n
walk relation for G



Albert R Meyer March 15, 2013

connected.4

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

length n walk relation

G itself is the length 1 walk relation: $G^1 = G$

lemma:

$$G^m \circ G^n = G^{m+n}$$

relational composition



Albert R Meyer March 15, 2013

connected.5

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

$$G^m \circ G^n = G^{m+n}$$

$$x(G^m \circ G^n)y ::= \exists z. x G^m z G^n y$$

IFF $x G^{m+n} y$

because a length $m+n$ walk consists of a length m walk followed by a length n walk



Albert R Meyer March 15, 2013

connected.6

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Matrices & Composition

$A_G ::=$ Adjacency matrix for G

Lemma: $A_{G \circ H} = A_H \odot A_G$

where \odot is Boolean matrix product—using OR instead of +



Albert R Meyer March 15, 2013

connected.8

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Matrices & Composition

So compute A_{G^n} by fast matrix exponentiation $\approx \log n$ matrix products.



Albert R Meyer March 15, 2013

connected.9

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Walk Relation

G^* is walk relation of G
 $u G^* v$ iff \exists walk u to v
 (u is connected to v)



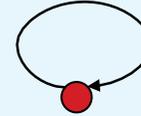
Albert R Meyer March 15, 2013

connected.10

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Compute the Walk Relation

Add self-loops:



$$G^{\leq} ::= G \cup \text{Id}_v$$

G^{\leq} has a length n walk iff
 G has a length $\leq n$ walk



Albert R Meyer March 15, 2013

connected.11

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

lengthening a walk in G

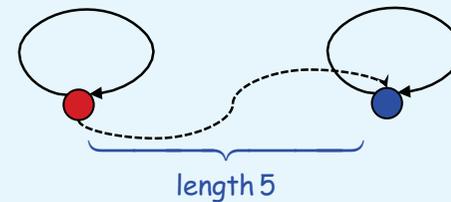


Albert R Meyer March 15, 2013

connected.12

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

lengthening a walk in G^{\leq}



just keep looping k times to
 make a length $5+k$ walk in G^{\leq}



Albert R Meyer March 15, 2013

connected.13

| | | | |
|----|---|----|----|
| 6 | 9 | 13 | 7 |
| 12 | | 10 | 5 |
| 3 | 1 | 4 | 14 |
| 15 | 8 | 11 | 2 |

Compute the Walk Relation

If G has n vertices, then
length of paths is $< n$, and

$$G^* = \left(G^{\leq}\right)^{n-1}$$

So find all connected vertex
pairs with $n^2 \log n$ AND/OR's



MIT OpenCourseWare
<http://ocw.mit.edu>

6.042J / 18.062J Mathematics for Computer Science
Spring 2015

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.