

## In-Class Problems Week 4, Fri.

### Problem 1.

Multiplying and dividing an integer  $n$  by 2 only requires a one digit left or right shift of the binary representation of  $n$ , which are hardware-supported fast operations on most computers. Here is a state machine,  $R$ , that computes the product of two nonnegative integers  $x$  and  $y$  using just these shift operations, along with integer addition:

$$\begin{aligned} \text{states} &::= \mathbb{N}^3 && \text{(triples of nonnegative integers)} \\ \text{start state} &::= (x, y, 0) \\ \text{transitions} &::= (r, s, a) \longrightarrow \begin{cases} (2r, s/2, a) & \text{for even } s > 0, \\ (2r, (s-1)/2, a+r) & \text{for odd } s > 0. \end{cases} \end{aligned}$$

(a) Verify that

$$P((r, s, a)) ::= [rs + a = xy] \tag{1}$$

is a preserved invariant of  $R$ .

(b) Prove that  $R$  is partially correct: if  $R$  reaches a final state—a state from which no transition is possible—then  $a = xy$ .

(c) Briefly explain why this state machine will terminate after a number of transitions bounded by a small constant times the *length* of the binary representation of  $y$ .

### Problem 2.

In this problem you will establish a basic property of a puzzle toy called the *Fifteen Puzzle* using the method of invariants. The Fifteen Puzzle consists of sliding square tiles numbered  $1, \dots, 15$  held in a  $4 \times 4$  frame with one empty square. Any tile adjacent to the empty square can slide into it.

The standard initial position is

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

We would like to reach the target position (known in the oldest author's youth as "the impossible"):

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	



A state machine model of the puzzle has states consisting of a  $4 \times 4$  matrix with 16 entries consisting of the integers  $1, \dots, 15$  as well as one “empty” entry—like each of the two arrays above.

The state transitions correspond to exchanging the empty square and an adjacent numbered tile. For example, an empty at position  $(2, 2)$  can exchange position with tile above it, namely, at position  $(1, 2)$ :

$n_1$	$n_2$	$n_3$	$n_4$
$n_5$		$n_6$	$n_7$
$n_8$	$n_9$	$n_{10}$	$n_{11}$
$n_{12}$	$n_{13}$	$n_{14}$	$n_{15}$

→

$n_1$		$n_3$	$n_4$
$n_5$	$n_2$	$n_6$	$n_7$
$n_8$	$n_9$	$n_{10}$	$n_{11}$
$n_{12}$	$n_{13}$	$n_{14}$	$n_{15}$

We will use the invariant method to prove that there is no way to reach the target state starting from the initial state.

We begin by noting that a state can also be represented as a pair consisting of two things:

1. a list of the numbers  $1, \dots, 15$  in the order in which they appear—reading rows left-to-right from the top row down, ignoring the empty square, and
2. the coordinates of the empty square—where the upper left square has coordinates  $(1, 1)$ , the lower right  $(4, 4)$ .

(a) Write out the “list” representation of the start state and the “impossible” state.

Let  $L$  be a list of the numbers  $1, \dots, 15$  in some order. A pair of integers is an *out-of-order pair* in  $L$  when the first element of the pair both comes *earlier* in the list and *is larger*, than the second element of the pair. For example, the list  $1, 2, 4, 5, 3$  has two out-of-order pairs:  $(4, 3)$  and  $(5, 3)$ . The increasing list  $1, 2, \dots, n$  has no out-of-order pairs.

Let a state,  $S$ , be a pair  $(L, (i, j))$  described above. We define the *parity* of  $S$  to be 0 or 1 depending on whether the sum of the number of out-of-order pairs in  $L$  and the row-number of the empty square is even or odd. that is

$$\text{parity}(S) ::= \begin{cases} 0 & \text{if } p(L) + i \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

(b) Verify that the parity of the start state and the target state are different.

(c) Show that the parity of a state is preserved under transitions. Conclude that “the impossible” is impossible to reach.

By the way, if two states have the same parity, then in fact there *is* a way to get from one to the other. If you like puzzles, you’ll enjoy working this out on your own.

### Problem 3.

The Massachusetts Turnpike Authority is concerned about the integrity of the new Zakim bridge. Their consulting architect has warned that the bridge may collapse if more than 1000 cars are on it at the same time. The Authority has also been warned by their traffic consultants that the rate of accidents from cars speeding across bridges has been increasing.

Both to lighten traffic and to discourage speeding, the Authority has decided to make the bridge *one-way* and to put tolls at *both* ends of the bridge (don’t laugh, this is Massachusetts). So cars will pay tolls both on entering and exiting the bridge, but the tolls will be different. In particular, a car will pay \$3 to enter onto the bridge and will pay \$2 to exit. To be sure that there are never too many cars on the bridge, the Authority will let a car onto the bridge only if the difference between the amount of money currently at the entry toll booth and the amount at the exit toll booth is strictly less than a certain threshold amount of  $\$T_0$ .

The consultants have decided to model this scenario with a state machine whose states are triples of nonnegative integers,  $(A, B, C)$ , where

- $A$  is an amount of money at the entry booth,
- $B$  is an amount of money at the exit booth, and
- $C$  is a number of cars on the bridge.

Any state with  $C > 1000$  is called a *collapsed* state, which the Authority dearly hopes to avoid. There will be no transition out of a collapsed state.

Since the toll booth collectors may need to start off with some amount of money in order to make change, and there may also be some number of “official” cars already on the bridge when it is opened to the public, the consultants must be ready to analyze the system started at *any* uncollapsed state. So let  $A_0$  be the initial number of dollars at the entrance toll booth,  $B_0$  the initial number of dollars at the exit toll booth, and  $C_0 \leq 1000$  the number of official cars on the bridge when it is opened. You should assume that even official cars pay tolls on exiting or entering the bridge after the bridge is opened.

(a) Give a mathematical model of the Authority’s system for letting cars on and off the bridge by specifying a transition relation between states of the form  $(A, B, C)$  above.

(b) Characterize each of the following derived variables

$$A, B, A + B, A - B, 3C - A, 2A - 3B, B + 3C, 2A - 3B - 6C, 2A - 2B - 3C$$

as one of the following

constant	C
strictly increasing	SI
strictly decreasing	SD
weakly increasing but not constant	WI
weakly decreasing but not constant	WD
none of the above	N

and briefly explain your reasoning.

The Authority has asked their engineering consultants to determine  $T$  and to verify that this policy will keep the number of cars from exceeding 1000.

The consultants reason that if  $C_0$  is the number of official cars on the bridge when it is opened, then an additional  $1000 - C_0$  cars can be allowed on the bridge. So as long as  $A - B$  has not increased by  $3(1000 - C_0)$ , there shouldn’t more than 1000 cars on the bridge. So they recommend defining

$$T_0 ::= 3(1000 - C_0) + (A_0 - B_0), \tag{2}$$

where  $A_0$  is the initial number of dollars at the entrance toll booth,  $B_0$  is the initial number of dollars at the exit toll booth.

(c) Use the results of part (b) to define a simple predicate,  $P$ , on states of the transition system which is satisfied by the start state—that is  $P(A_0, B_0, C_0)$  holds—is not satisfied by any collapsed state, and is a preserved invariant of the system. Explain why your  $P$  has these properties. Conclude that the traffic won’t cause the bridge to collapse.

(d) A clever MIT intern working for the Turnpike Authority agrees that the Turnpike’s bridge management policy will be *safe*: the bridge will not collapse. But she warns her boss that the policy will lead to *deadlock*—a situation where traffic can’t move on the bridge even though the bridge has not collapsed.

Explain more precisely in terms of system transitions what the intern means, and briefly, but clearly, justify her claim.

**Supplemental problem:**

**Problem 4.**

A classroom is designed so students sit in a square arrangement. An outbreak of beaver flu sometimes infects students in the class; beaver flu is a rare variant of bird flu that lasts forever, with symptoms including a yearning for more quizzes and the thrill of late night problem set sessions.

Here is an illustration of a  $6 \times 6$ -seat classroom with seats represented by squares. The locations of infected students are marked with an asterisk.

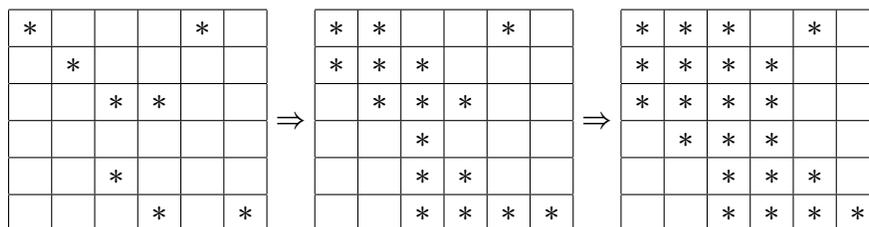
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Outbreaks of infection spread rapidly step by step. A student is infected after a step if either

- the student was infected at the previous step (since beaver flu lasts forever), or
- the student was adjacent to *at least two* already-infected students at the previous step.

Here *adjacent* means the students' individual squares share an edge (front, back, left or right); they are not adjacent if they only share a corner point. So each student is adjacent to 2, 3 or 4 others.

In the example, the infection spreads as shown below.



In this example, over the next few time-steps, all the students in class become infected.

**Theorem.** *If fewer than  $n$  students among those in an  $n \times n$  arrangement are initially infected in a flu outbreak, then there will be at least one student who never gets infected in this outbreak, even if students attend all the lectures.*

Prove this theorem.

*Hint:* Think of the state of an outbreak as an  $n \times n$  square above, with asterisks indicating infection. The rules for the spread of infection then define the transitions of a state machine. Find a weakly decreasing derived variable that leads to a proof of this theorem. (If you don't see it within 4 minutes, ask your TA for a four word hint.)

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