Problem Set 7 Solutions

Due: Monday, April 4 at 9 PM

Problem 1. Every function has some subset of these properties:

injective surjective bijective

Determine the properties of the functions below, and briefly explain your reasoning.

(a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.

Solution. This function is injective, since e^x takes on each nonnegative real value for exactly one x. However, the function is not surjective, because e^x never takes on negative values. Therefore, the function is not bijective either.

(b) The function $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x$.

Solution. The function e^x takes on every nonnegative value for exactly one x, so it is injective, surjective, and bijective.

(c) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x + 1)x(x - 1)$.

Solution. This function is surjective, since it is continuous, it tends to $+\infty$ for large positive x, and tends to $-\infty$ for large negative x. Thus, the function takes on each real value for at least one x. However, this function is not injective, since it takes on the value 0 at $x = -1$, $x = 0$, and $x = 1$. Therefore, the function is not bijective either.

(d) Let S be the set of all 20-bit sequences. Let T be the set of all 10-bit sequences. Let $f : S \to T$ map each 20-bit sequence to its first 10 bits. For example:

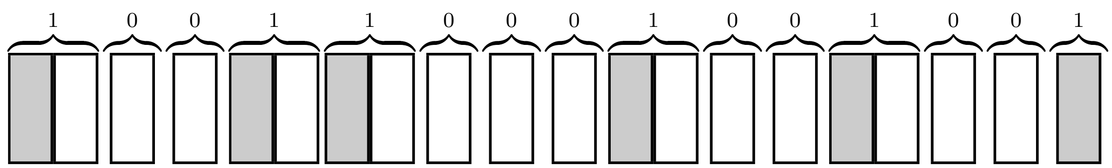
$$f(\underline{1111011010}1101010010) = 1111011010$$

Solution. This function is surjective since the sequence $b_1 b_2 \ldots b_{10}$ is mapped to by $b_1 b_2 \ldots b_{10} 00 \ldots 0$, for example. However, the function is not injective, because this sequence is also mapped to by $b_1 b_2 \ldots b_{10} 11 \ldots 1$. Consequently, the function is not bijective either.

Problem 2. There are 20 books arranged in a row on a shelf.

(a) Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15-bit sequences with exactly 6 ones.

Solution. There is a bijection from 15-bit sequences with exactly six 1's to valid book selections: given such a sequence, map each zero to a non-chosen book, each of the first five 1's to a chosen book followed by a non-chosen book, and the last 1 to a chosen book. For example, here is a configuration of books and the corresponding binary sequence:



Selected books are darkened. Notice that the first fives ones are mapped to a chosen book *and* an non-chosen book in order to ensure that the binary sequence maps to a valid selection of books.

(b) How many ways are there to select 6 books so that no two adjacent books are selected?

Solution. By the Bijection Rule, this is equal to the number of 15-bit sequences with exactly 6 ones, which is equal to

$$\frac{15!}{6!\,9!} = \binom{15}{6}$$

by the Bookkeeper Rule.

Problem 3. Answer the following questions and provide brief justifications. Not every problem can be solved with a cute formula; you may have to fall back on case analysis, explicit enumeration, or ad hoc methods.

You may leave factorials and binomial coefficients in your answers.

(a) In how many different ways can the letters in the name of the popular 1980's band $BANANARAMA$ be arranged?

Solution. There are 5 A's, 2 N's, 1 B, 1 R, and 1 M. Therefore, the number of arrangements is

$$\frac{10!}{5!\,2!\,1!\,1!\,1!}$$

by the Bookkeeper Rule.

(b) How many different paths are there from point $(0, 0, 0)$ to point $(12, 24, 36)$ if every step increments one coordinate and leaves the other two unchanged?

Solution. There is a bijection between the set of all such paths and the set of strings containing 12 X's, 24 Y's, and 36 Z's. In particular, we obtain a path by working through a string from left to right. An X corresponds to a step that increments the first coordinate, a Y increments the second coordinate, and a Z increments the third. The number of such strings is:

$$\frac{72!}{12!\ 24!\ 36!}$$

Therefore, this is also the number of paths.

(c) In how many different ways can $2n$ students be paired up?

Solution. Pair up students by the following procedure. Line up the students and pair the first and second, the third and fourth, the fifth and sixth, etc. The students can be lined up in $(2n)!$ ways. However, this overcounts by a factor of 2^n, because we would get the same pairing if the first and second students were swapped, the third and fourth were swapped, etc. Furthermore, we are still overcounting by a factor of $n!$, because we would get the same pairing even if pairs of students were permuted, e.g. the first and second were swapped with the ninth and tenth. Therefore, the number of pairings is:

$$\frac{(2n)!}{2^n \cdot n!}$$

(d) How many different solutions over the natural numbers are there to the following equation?

$$x_1 + x_2 + x_3 + \ldots + x_8 = 100$$

A solution is a specification of the value of each variable x_i. Two solutions are different if different values are specified for some variable x_i.

Solution. There is a bijection between sequences containing 100 zeros and 7 ones. Specifically, the 7 ones divide the zeros into 8 segments. Let x_i be the number of zeros in the i-th segment. Therefore, the number of solutions is:

$$\binom{100+7}{7}$$

(e) In how many different ways can one choose n out of $2n$ objects, given that n of the $2n$ objects are identical and the other n are all unique?

Solution. We can select n objects as follows. First, take a subset of the unique objects. Then take however many identical elements are needed to bring the total to n. The first step can be done in 2^n ways, and the second can be done in only 1 way. Therefore, there are 2^n ways to choose n objects.

(f) How many undirected graphs are there with vertices $v_1, v_2, \ldots, v_n$ if self-loops are permitted?

Solution. There are $\binom{n}{2} + n$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{\binom{n}{2}+n}$$

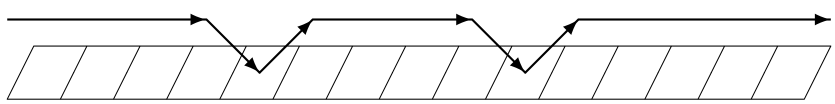
(g) In how many different ways can 10 indistinguishable balls be placed in four distinguishable boxes, such that every box gets 1, 2, 3, or 4 balls?

Solution. First, we might as well put 1 ball in every box. Now the problem is to put the remaining 6 balls into 4 boxes so that no box gets more than 3 balls. Now we turn to case analysis. For example, we could put 3 balls in two boxes and 0 balls in the other two boxes. There are $\frac{4!}{2!\,2!} = 6$ ways to do this. All cases are listed below:

distribution of balls	# of ways
$3, 3, 0, 0$	$\frac{4!}{2!\,2!} = 6$
$3, 2, 1, 0$	$\frac{4!}{1!\,1!\,1!\,1!} = 24$
$3, 1, 1, 1$	$\frac{4!}{3!\,1!} = 4$
$2, 2, 2, 0$	$\frac{4!}{3!\,1!} = 4$
$2, 2, 1, 1$	$\frac{4!}{2!\,2!} = 6$

Thus, there are $6 + 24 + 4 + 4 + 6 = 44$ ways in all.

(h) There are 15 sidewalk squares in a row. Suppose that a ball can be thrown so that it bounces on 0, 1, 2, or 3 distinct sidewalk squares. Assume that the ball always moves from left to right. How many different throws are possible? As an example, a two-bounce throw is illustrated below.



Solution.

$$\binom{15}{0} + \binom{15}{1} + \binom{15}{2} + \binom{15}{3}$$

(i) In how many different ways can the numbers shown on a red die, a green die, and a blue die total up to 15? Assume that these are ordinary 6-sided dice.

Solution. We fall back on explicit enumeration. Let R, G, and B be the values shown on the three dice.

$$\begin{aligned}
R = 1, B + G = 14 &\rightarrow \text{0 ways}\\
R = 2, B + G = 13 &\rightarrow \text{0 ways}\\
R = 3, B + G = 12 &\rightarrow \text{1 way}\\
R = 4, B + G = 11 &\rightarrow \text{2 ways}\\
R = 5, B + G = 10 &\rightarrow \text{3 ways}\\
R = 6, B + G = 9 &\rightarrow \text{4 ways}
\end{aligned}$$

(j) The working days in the next year can be numbered 1, 2, 3, ..., 300. I'd like to avoid as many as possible.

- On even-numbered days, I'll say I'm sick.
- On days that are a multiple of 3, I'll say I was stuck in traffic.
- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I *avoid* in the coming year?

Solution. Let D_2 be the set of even-numbered days, D_3 be the days that are a multiple of 3, and D_5 be days that are a multiple of 5. The set of days I can avoid is $D_2 \cup D_3 \cup D_5$. By the Inclusion-Exclusion Rule, the size of this set is:

$$\begin{aligned} |D_2 \cup D_3 \cup D_5| &= |D_2| + |D_3| + |D_5| \\ &\quad - |D_2 \cap D_3| - |D_2 \cap D_5| - |D_3 \cap D_5| \\ &\quad + |D_2 \cap D_3 \cap D_5| \\ &= \frac{300}{2} + \frac{300}{3} + \frac{300}{5} - \frac{300}{2 \cdot 3} - \frac{300}{2 \cdot 5} - \frac{300}{3 \cdot 5} + \frac{300}{2 \cdot 3 \cdot 5} \\ &= 220 \end{aligned}$$

Problem 4. Use the pigeonhole principle to solve the following problems.

(a) Prove that among any $n^2 + 1$ points within an $n \times n$ square there must exist two points whose distance is at most $\sqrt{2}$.

Solution. Partition the $n \times n$ into n^2 unit squares. Each of the $n^2 + 1$ points lies in one of these n^2 unit squares. (If a point lies on the boundary between squares, assign it to a square arbitrarily.) Therefore, by the pigeonhole principle, there exist two points in the same unit square. And the distance between those two points can be at most $\sqrt{2}$.

(b) Jellybeans of 6 different flavors are stored in 5 jars. There are 11 jellybeans of each flavor. Prove that some jar contains at least three jellybeans of one flavor and also at least three jellybeans of some other flavor.

Solution. We use the pigeonhole principle twice. Since there are 11 beans of a given flavor and there are only 5 jars, some jar must get at least $\rceil 11/5 \lceil = 3$ beans of that flavor. Since there are 6 flavors and only 5 jars, some jar must get two pairs of same-flavored beans.

(c) Prove that among every set of 30 integers, there exist two whose difference *or* sum is a multiple of 51.

Solution. Regard the 30 integers as pigeons. Regard the 26 sets $\{0\}$, $\{1, 50\}$, $\{2, 49\}$, ..., $\{25, 26\}$ as pigeonholes. Map integer n to the set containing $n \text{ rem } 51$. By the pigeonhole principle, some two pigeons (integers a and b) are mapped to the same pigeonhole. Thus, either:

- $a \text{ rem } 51 = b \text{ rem } 51$ and so the difference of a and b is a multiple of 51.

- $a \text{ rem } 51 + b \text{ rem } 51$ is either 0 or 51 and so the sum of a and b is a multiple of 51.

Problem 5. A *balanced parentheses string* is a sequence of left and right parentheses such that

- the total number of left and right parentheses is equal, and
- the number of left parentheses is greater than or equal to the number of right parentheses in every prefix.

For example:

Balanced	Not Balanced	
(())	((()	more left than right
()(())	())()(	fewer left than right in prefix ())

Let C_n be the number of balanced parentheses strings with $2n$ parentheses. The values $C_0, C_1, C_2, \ldots$ are the *Catalan numbers*, which come up in dozens of different counting problems. Here are the first few Catalan numbers:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012

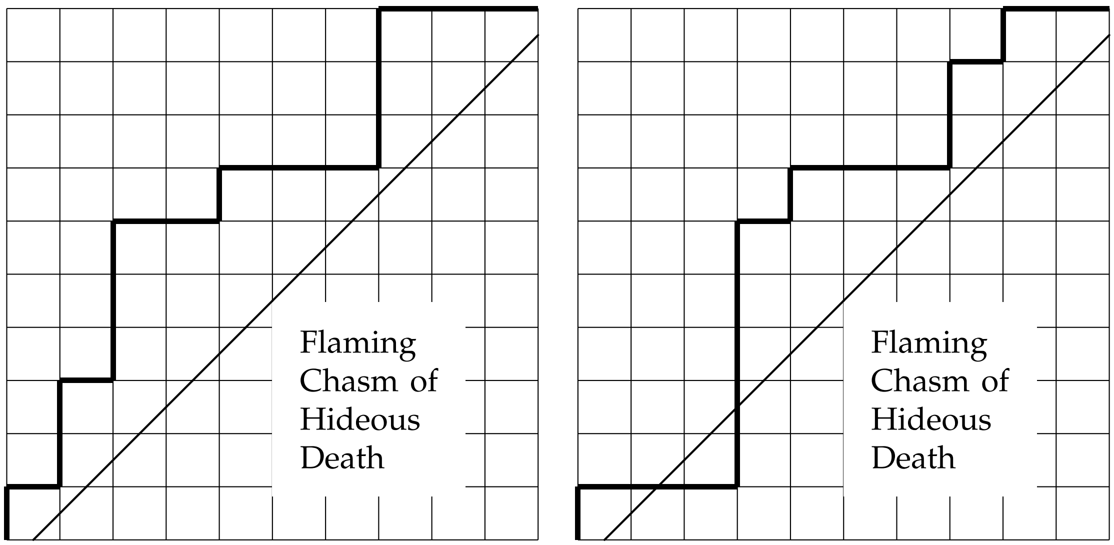
(a) Verify the first four entries by listing all balanced parentheses strings with at most 6 parentheses. (Don't forget the empty string!)

Solution. Here are all the balanced parentheses strings with at most 6 parentheses:

empty	()	()()	(())	
()(())	(())()	(()())	((()))	()()()

Therefore, $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, and $C_3 = 5$ as claimed in the table.

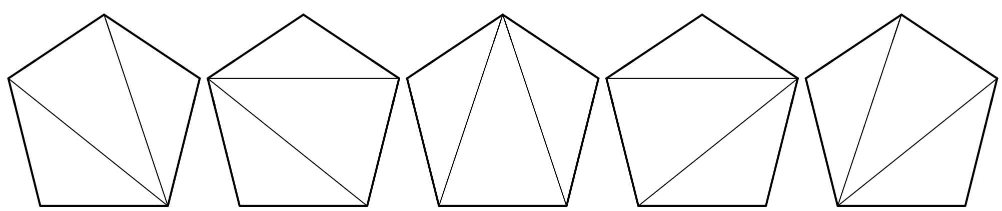
(b) A path from $(0, 0)$ to (n, n) consisting of unit steps upward or rightward is *safe* if it does not cross the diagonal boundary of the Flaming Chasm of Hideous Death.



A Safe Path An Unsafe Path

Show that there are exactly C_n safe paths by describing a bijection with balanced parentheses strings.

(c) Many computational geometry algorithms begin by partitioning polygons into triangles with the same vertices. For example, here are all the possible triangulations of a pentagon:

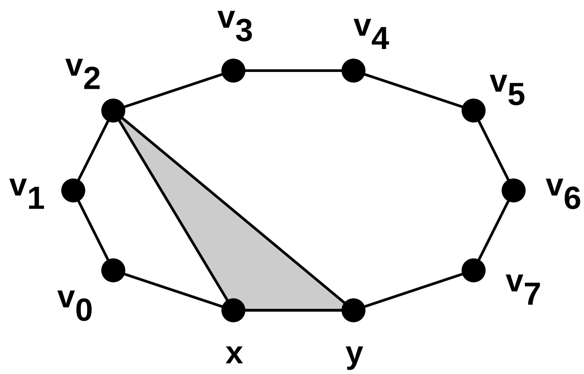


Show that there are C_n different ways to triangulate a convex $(n+2)$-gon by describing a bijection from triangulations to balanced parentheses strings. (This problem is challenging; ask your TA for help if you're stuck!)

Solution. Note that every nonempty balanced parentheses string has the form $(\alpha)\beta$ since the first symbol must be (and there must be a matching). Thus, every balanced parentheses string can be decomposed into two smaller strings α and β. We'll exploit this fact to recursively map a balanced parentheses string to a triangulation.

Suppose we have a not-yet-triangulated $(n+2)$-gon and a balanced parentheses string $(\alpha)\beta$. We must map the parentheses string to a triangulation.

Select two consecutive vertices x and y and denote the remaining vertices $v_0, \ldots, v_{n-1}$. It must be that α contains j pairs of parentheses and β contains the remaining $(n-1)-j$ pairs for some j between 0 and $n-1$. Draw a triangle with the vertices x, y, and v_j.



This triangle partitions the polygon into a $(j+2)$-gon and a $(n-j+1)$-gon. Triangulate these polygons recursively using the balanced parentheses strings α and β. (Let x and v_j play the role of x and y in triangulation one side, and let v_j and y play the role of x and y on the other.)

Problem 6. A *derangement* is a permutation $(x_1, x_2, \ldots, x_n)$ of the set $\{1, 2, \ldots, n\}$ such that $x_i \neq i$ for all i. For example, $(2, 3, 4, 5, 1)$ is a derangement, but $(2, 1, 3, 5, 4)$ is not because 3 appears in the third position. The objective of this problem is to count derangements.

(a) Let's first work on counting permutations of $\{1, 2, \ldots, n\}$ that are *not* derangements. Let S_i be the set of all permutations $(x_1, x_2, \ldots, x_n)$ of the set $\{1, 2, \ldots, n\}$ such that $x_i = i$. Use the inclusion-exclusion formula to express the number of non-derangements in terms of sizes of intersections of the sets $S_1, \ldots, S_n$.

Solution.

$$\sum_i |S_i| - \sum_{i,j} |S_i \cap S_j| + \sum_{i,j,k} |S_i \cap S_j \cap S_k| - \ldots \pm |S_1 \cap S_2 \cap \ldots \cap S_n|$$

In each summation, the subscripts are distinct elements of $\{1, \ldots, n\}$.

(b) What is $|S_i|$?

Solution. There is a bijection between permutations of $\{1, 2, \ldots, n\}$ with i in the i-th position and unrestricted permutations of $\{1, 2, \ldots, n\} - i$. Therefore, $|S_i| = (n-1)!$.

(c) What is $|S_i \cap S_j|$ where $i \neq j$?

Solution. The set $S_i \cap S_j$ consists of all permutations with i in the i-th position and j in the j-th position. Thus, there is a bijection with permutations of $\{1, 2, \ldots, n\} - \{i, j\}$, and so $|S_i \cap S_j| = (n-2)!$.

(d) What is $|S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k}|$ where $i_1, i_2, \ldots, i_k$ are all distinct?

Solution. By the same argument, $(n-k)!$.

(e) How many terms in the expression in part (a) have the form $|S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k}|$?

Solution. There is one such term for each k-element subset of the n-element set $\{1, 2, \ldots, n\}$. Therefore, there are $\binom{n}{k}$ such terms.

(f) Combine your answers to the preceding parts to get a (messy) expression for the number of *non*-derangements.

Solution.

$$\begin{aligned}
&\sum_i |S_i| - \sum_{i,j} |S_i \cap S_j| + \sum_{i,j,k} |S_i \cap S_j \cap S_k| - \ldots \pm |S_1 \cap S_2 \cap \ldots \cap S_n| \\
&= \binom{n}{1} \cdot (n-1)! - \binom{n}{2} \cdot (n-2)! + \binom{n}{3} \cdot (n-3)! - \ldots \pm \binom{n}{n} \cdot 0! \\
&= n! \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots \pm \frac{1}{n!} \right)
\end{aligned}$$

(g) Now give an expression for the number of derangements.

Solution.

$$n! \left(1 - \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots \pm \frac{1}{n!} \right)$$

(h) As n goes to infinity, this expression approaches a constant fraction of all permutations. What is that constant? Recall that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Solution. $1/e$