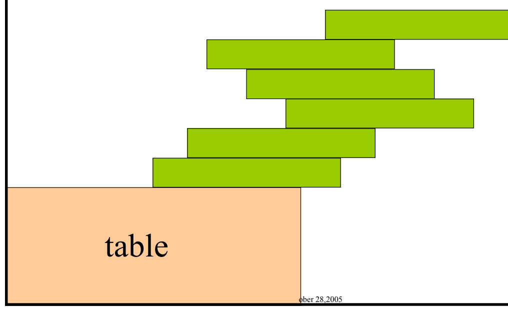
Mathematics for Computer Science

MIT 6.042J/18.062J

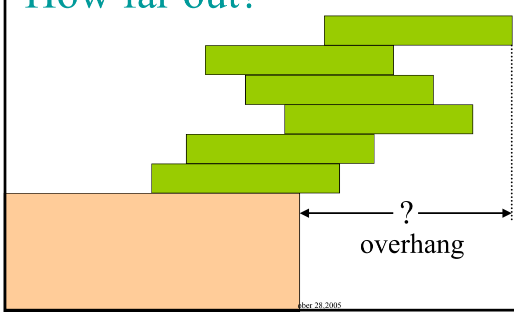
Harmonic Series, Integral Method, Stirling's Formula

Book Stacking



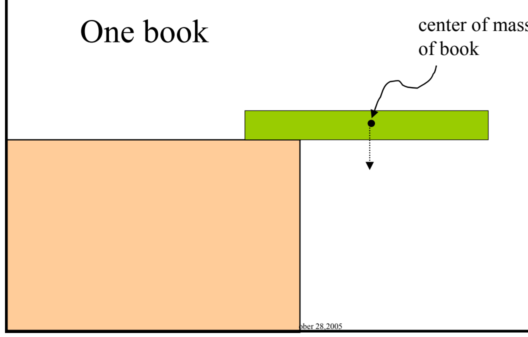
Book Stacking

How far out?



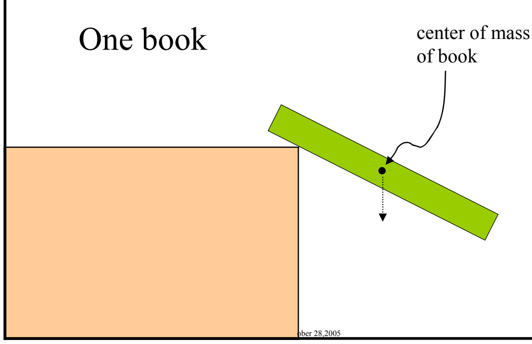
Book Stacking

One book



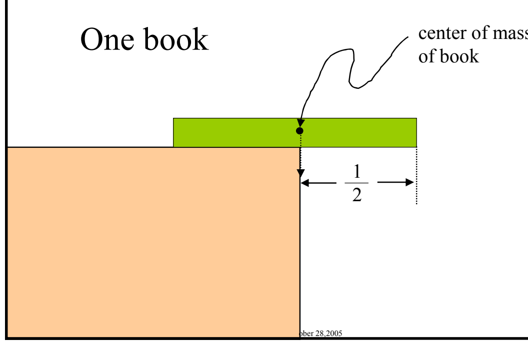
Book Stacking

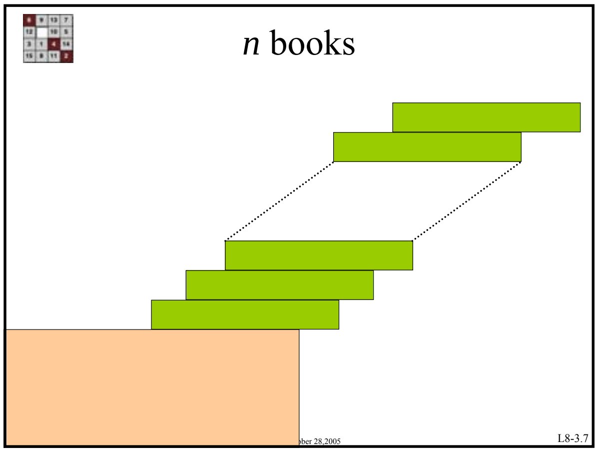
One book

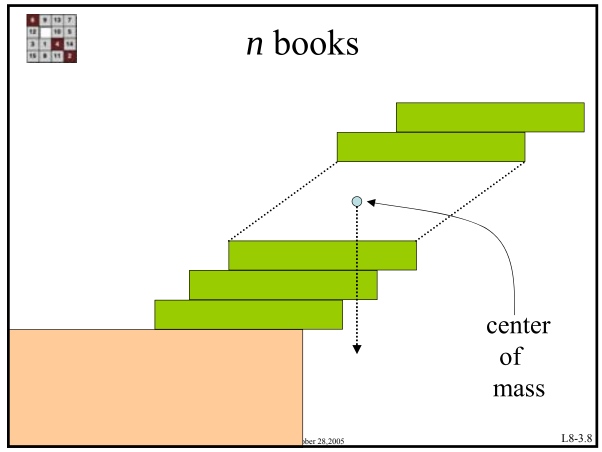


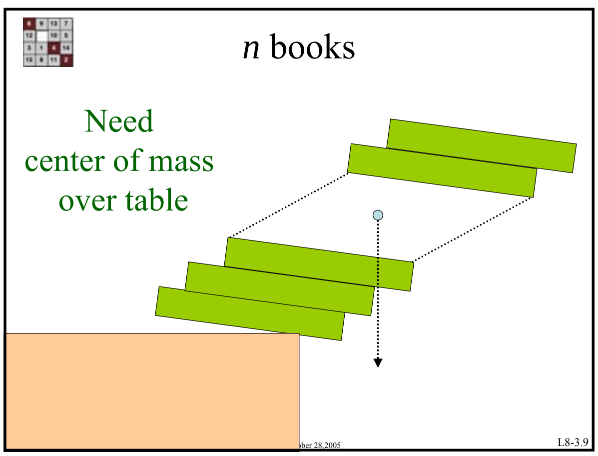
Book Stacking

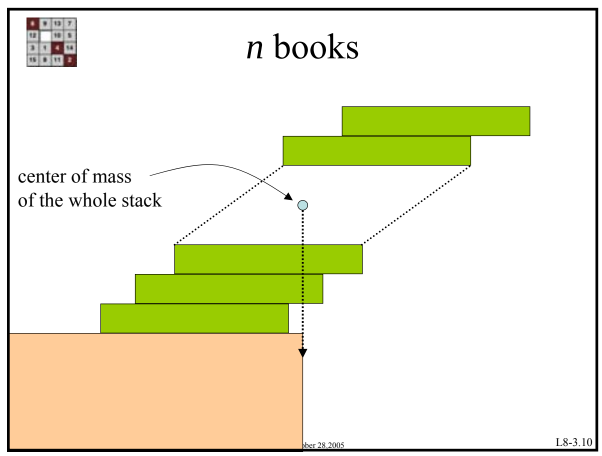
One book

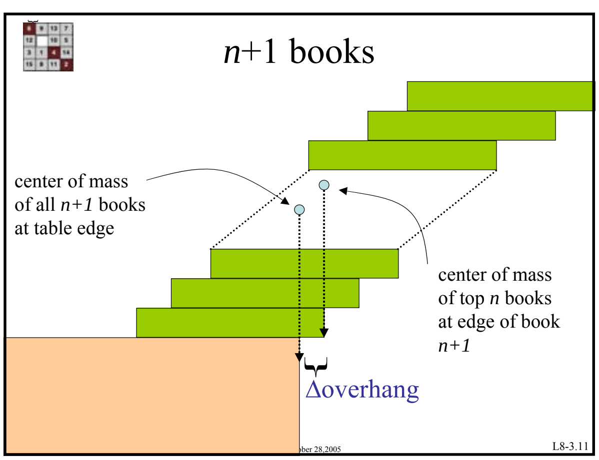








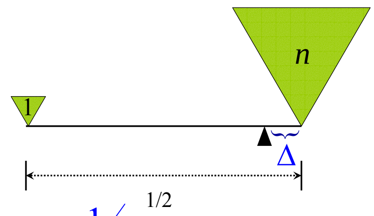




Δ overhang ::=
Horizontal distance from
n-book to *n*+1-book
centers-of-mass

October 28,2005
L8-3.12

Δ overhang



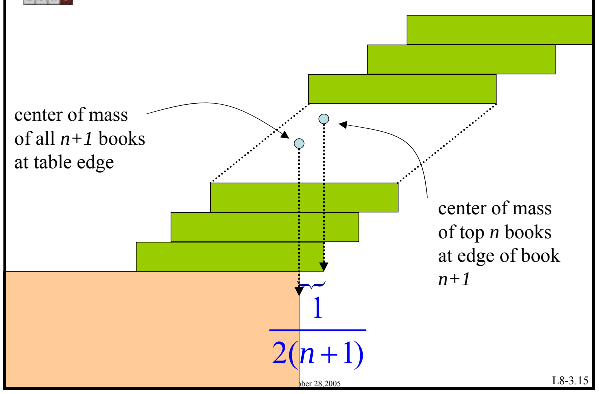
$$\Delta = \frac{1/2}{n+1} = \frac{1}{2(n+1)}$$

Choose origin so center of n-stack at $x = 0$.

Now center of n+1[st] book is at $x = 1/2$, so center of n+1-stack is at

$$x = \frac{n \cdot 0 + 1 \cdot 1/2}{n+1} = \frac{1}{2(n+1)}$$

n+1 books



Book stacking summary

B_n ::= overhang of n books

$B_1 = 1/2$

$$B_{n+1} = B_n + \frac{1}{2(n+1)}$$

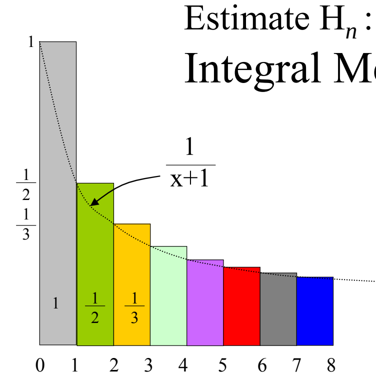
$$B_n = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

n[th] *Harmonic number*

$$B_n = H_n/2$$

Estimate H_n: Integral Method



$$\int_0^n \frac{1}{x+1}\,dx \;\le\; 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

$$\int_1^{n+1} \frac{1}{x}\,dx \;\le\; H_n$$

$$\ln(n+1) \;\le\; H_n$$

Book stacking

So $H_n \to \infty$ as $n \to \infty$, and so

overhang can be any desired size.

Book stacking

Overhang 3: need $B_n \ge 3$

$H_n \ge 6$

Integral bound: $\ln(n+1) \ge 6$

So can do with $n \ge \lceil e^6 - 1 \rceil$ = 403 books

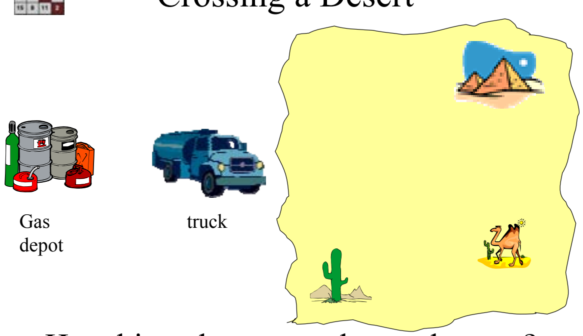
Actually calculate H_n:

227 books are enough.

Team Problem

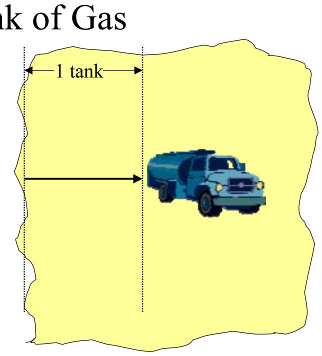
Problem 1

Crossing a Desert



How big a desert can the truck cross?

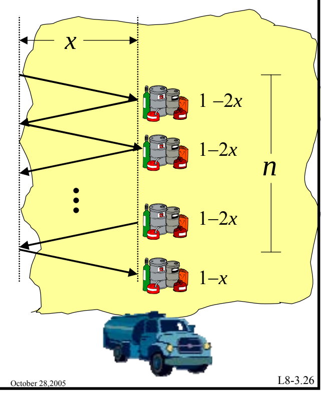
1 Tank of Gas



D_1::= max distance on 1 tank = 1

Let D_n ::=
max distance into the desert using *n* tanks of gas from the depot

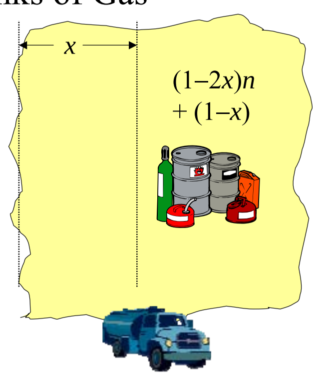
n+1 Tanks of Gas



n+1 Tanks of Gas

So have:

Set depot at *x* to be *n* tanks; continue from *x* with *n* tank method.



depot at *x*

Set $\underbrace{(1-2x)n + (1-x)} = n$.

Then using *n* tank strategy from position *x*, gives

$$\mathrm{D}_{n+1} = \mathrm{D}_n + x$$

$$(1-2x)n + (1-x) = n$$

$$x = \frac{1}{2n+1}$$

$$\boxed{\mathrm{D}_{n+1} = \mathrm{D}_n + \frac{1}{2n+1}}$$

$$\mathrm{D}_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

$$\int_0^n \frac{1}{2(x+1)-1}\,dx \le \mathrm{D}_n$$

$$\frac{\ln(2n+1)}{2} \le \mathrm{D}_n$$

Can cross any desert!

Team Problem

Problem 2

Closed form for *n*!

Factorial defines a product:

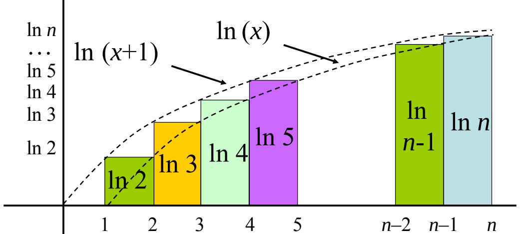
$$n! \;::=\; 1\cdot 2\cdot 3\cdots(n-1)\cdot n \;=\prod_{i=1}^{n} i$$

Turn product into a sum taking logs:

$$\ln(n!) = \ln(1\cdot 2\cdot 3\cdots(n-1)\cdot n)$$
$$= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$
$$= \sum_{i=1}^{n} \ln(i)$$

Integral Method

Integral Method to bound $\sum_{i=1}^{n} \ln i$



Integral Method

Bounds on ln(*n*!)

$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^{n} \ln(i) \le \int_1^n \ln(x+1)\,dx$$

Integral Method

Reminder:

$$\int \ln x\,dx = x\ln\left(\frac{x}{e}\right) =$$

$$x(\ln x - \ln e) = x(\ln x - 1)$$
$$= x\ln x - x$$

Integral Method

$$\int \ln x\,dx = x\ln\left(\frac{x}{e}\right)$$
$$= x\ln x - x$$

Quickie:
verify by differentiating.

Integral Method

Bounds on $\ln(n!)$

$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^{n} \ln(i) \le \int_1^n \ln(x+1)\,dx$$

$$n\ln\left(\frac{n}{e}\right)+1 \le \sum_{i=1}^{n} \ln(i) \le (n+1)\cdot\ln\left(\frac{n+1}{e}\right)+0.6$$

So guess: $\sum_{i=1}^{n} \ln(i) \approx (n+\frac{1}{2})\ln\left(\frac{n}{e}\right)$

Integral Method

$$\sum_{i=1}^{n} \ln(i) \approx (n+\frac{1}{2})\ln\left(\frac{n}{e}\right)$$

exponentiating:

$$n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^n$$

Stirling's Formula

A precise approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Asymptotic Equivalence

$$f(n) \sim g(n)$$

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 1$$

Team Problem

Problem 3

