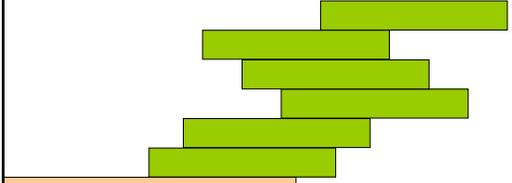



Mathematics for Computer Science
 MIT 6.042J/18.062J

Harmonic Series, Integral Method, Stirling's Formula

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Book Stacking

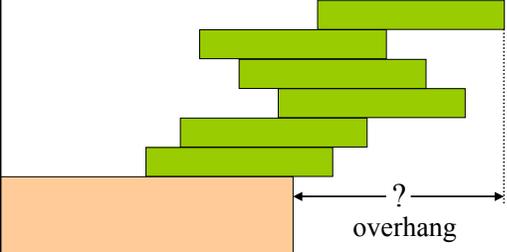


table

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Book Stacking

How far out?

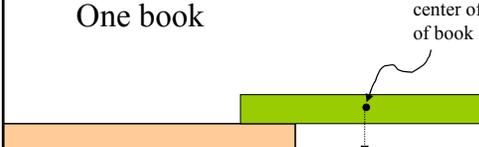


overhang

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Book Stacking

One book

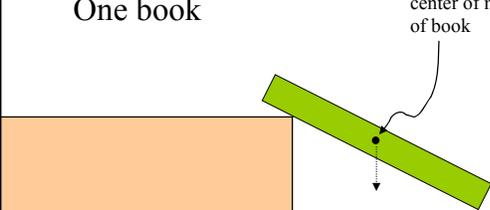


center of mass of book

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Book Stacking

One book

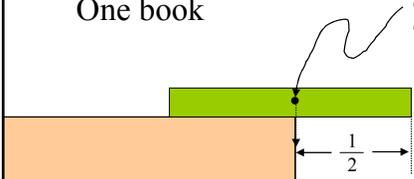


center of mass of book

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Book Stacking

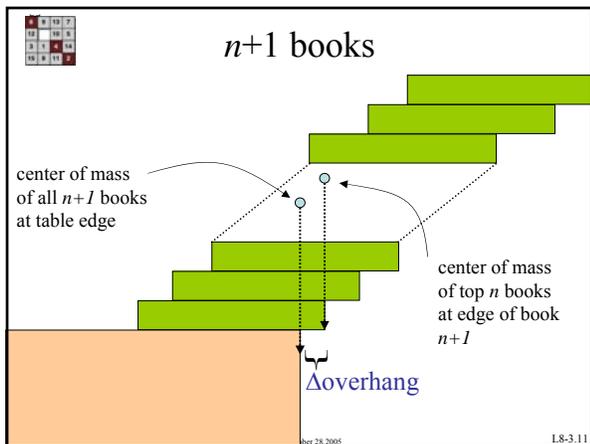
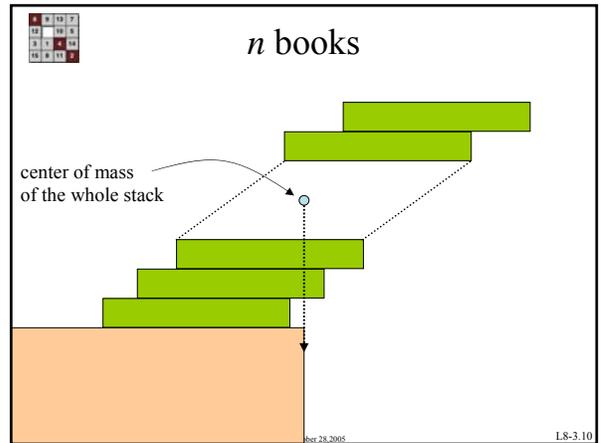
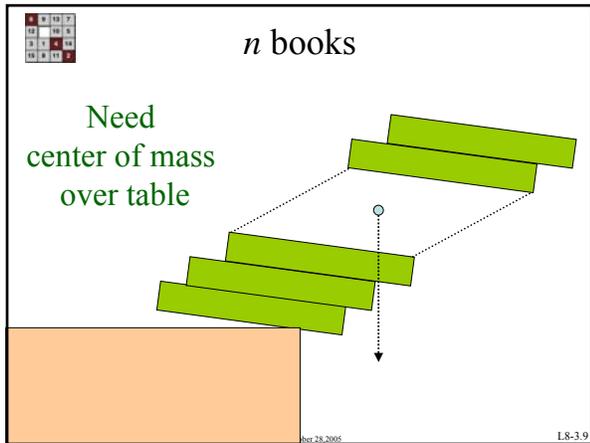
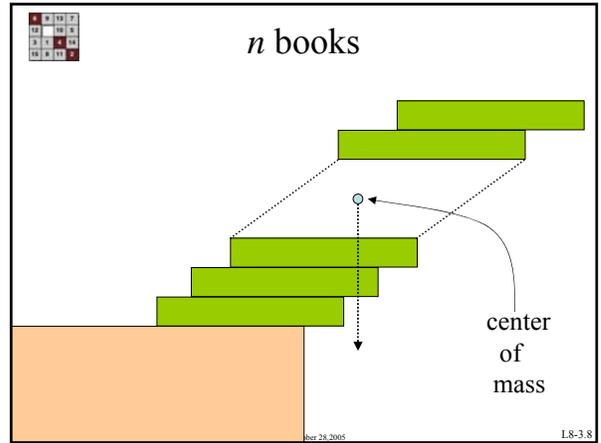
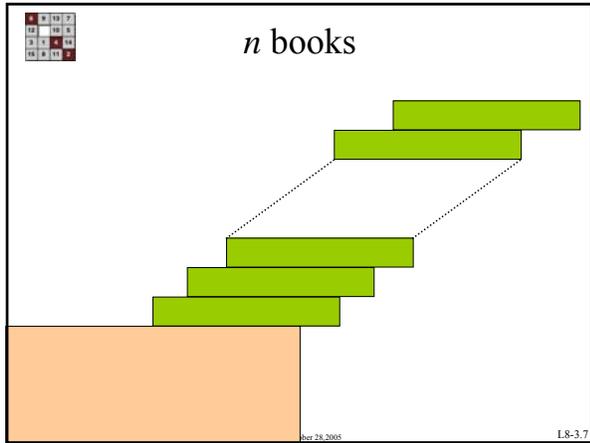
One book



center of mass of book

$\frac{1}{2}$

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Δ overhang ::=

Horizontal distance from n -book to $n+1$ -book centers-of-mass

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Δ overhang

$$\Delta = \frac{1/2}{n+1} = \frac{1}{2(n+1)}$$

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Choose origin so center of n -stack at $x = 0$.
 Now center of $n+1$ st book is at $x = 1/2$, so
 center of $n+1$ -stack is at

$$x = \frac{n \cdot 0 + 1 \cdot 1/2}{n+1} = \frac{1}{2(n+1)}$$

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$n+1$ books

center of mass of all $n+1$ books at table edge

center of mass of top n books at edge of book $n+1$

$$\frac{1}{2(n+1)}$$

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Book stacking summary

$B_n ::=$ overhang of n books

$$B_1 = 1/2$$

$$B_{n+1} = B_n + \frac{1}{2(n+1)}$$

$$B_n = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right)$$

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$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

n^{th} Harmonic number

$$B_n = H_n/2$$

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**Estimate H_n :
Integral Method**

$\frac{1}{x+1}$

0 1 2 3 4 5 6 7 8

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$$\int_0^n \frac{1}{x+1} dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$\int_1^{n+1} \frac{1}{x} dx \leq H_n$$

$$\ln(n+1) \leq H_n$$

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Book stacking

So $H_n \rightarrow \infty$ as $n \rightarrow \infty$, and so overhang can be any desired size.

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Book stacking

Overhang 3: need $B_n \geq 3$
 $H_n \geq 6$

Integral bound: $\ln(n+1) \geq 6$
So can do with $n \geq \lceil e^6 - 1 \rceil = 403$ books
Actually calculate H_n :
227 books are enough.

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Team Problem

Problem 1

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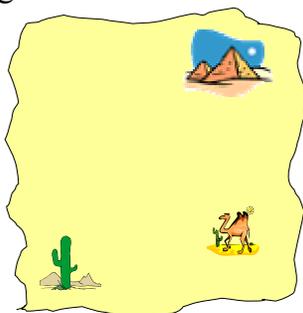
Crossing a Desert



Gas depot



truck



How big a desert can the truck cross?

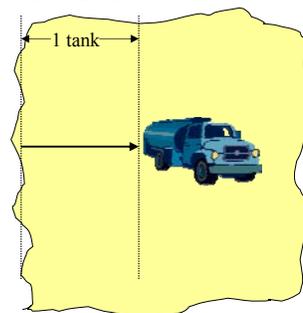
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1 Tank of Gas



$D_1 ::= \max \text{ distance on 1 tank} = 1$

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L.8-3.24



Let $D_n ::=$
 max distance into the
 desert using n tanks
 of gas from the depot

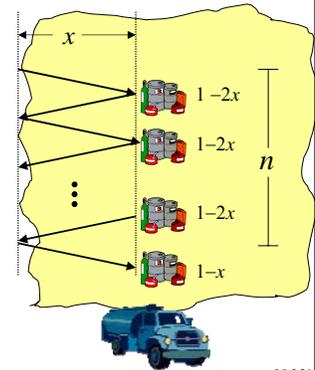
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$n+1$ Tanks of Gas



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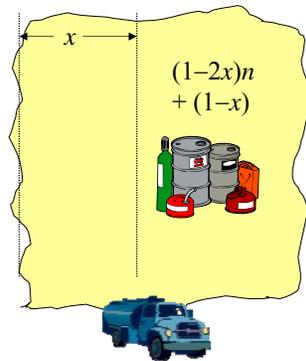
I.8-3.26



$n+1$ Tanks of Gas

So have:

Set depot at x
 to be n tanks;
 continue from
 x with n tank
 method.



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depot at x



$$\text{Set } (1-2x)n + (1-x) = n.$$

Then using n tank strategy
 from position x , gives

$$D_{n+1} = D_n + x$$

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$$(1-2x)n + (1-x) = n$$

$$x = \frac{1}{2n+1}$$

$$D_{n+1} = D_n + \frac{1}{2n+1}$$

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$$D_n = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}$$

$$\int_0^n \frac{1}{2(x+1)-1} dx \leq D_n$$

$$\frac{\ln(2n+1)}{2} \leq D_n$$

Can cross any desert!

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Team Problem

Problem 2

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Closed form for $n!$

Factorial defines a **product**:

$$n! ::= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n = \prod_{i=1}^n i$$

Turn product into a **sum** taking logs:

$$\begin{aligned} \ln(n!) &= \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) \\ &= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n) \\ &= \sum_{i=1}^n \ln(i) \end{aligned}$$

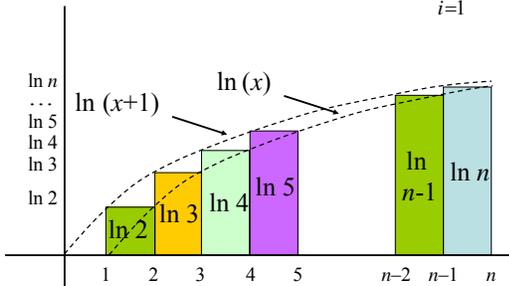
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Integral Method

Integral Method to bound $\sum_{i=1}^n \ln i$



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Integral Method

Bounds on $\ln(n!)$

$$\int_1^n \ln(x) dx \leq \sum_{i=1}^n \ln(i) \leq \int_1^n \ln(x+1) dx$$

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Integral Method

Reminder:

$$\int \ln x dx = x \ln \left(\frac{x}{e} \right) =$$

$$\begin{aligned} x(\ln x - \ln e) &= x(\ln x - 1) \\ &= x \ln x - x \end{aligned}$$

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Integral Method

$$\begin{aligned} \int \ln x dx &= x \ln \left(\frac{x}{e} \right) \\ &= x \ln x - x \end{aligned}$$

Quickie:
verify by differentiating.

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Integral Method

Bounds on $\ln(n!)$

$$\int_1^n \ln(x) dx \leq \sum_{i=1}^n \ln(i) \leq \int_1^{n+1} \ln(x+1) dx$$

$$n \ln\left(\frac{n}{e}\right) + 1 \leq \sum_{i=1}^n \ln(i) \leq (n+1) \cdot \ln\left(\frac{n+1}{e}\right) + 0.6$$

So guess:
$$\sum_{i=1}^n \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$$

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Integral Method

$$\sum_{i=1}^n \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$$

exponentiating:

$$n! \approx \sqrt{n/e} \left(\frac{n}{e}\right)^n$$

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Stirling's Formula

A precise approximation:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

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I.S-3.39



Asymptotic Equivalence

$$f(n) \sim g(n)$$

$$\lim_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)}\right) = 1$$

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I.S-3.40



Team Problem

Problem 3

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