Proofs by Induction

An Example of Induction

Suppose we have a property (say *color*) of the natural numbers:

0, 1, 2, 3, 4, 5, …

Showing that *zero is red*, and that the *successor of any red number is red*,

proves that *all numbers are red*!

The Induction Rule

0 and (from n to n+1)

proves 0, 1, 2, 3,….

$$\frac{R(0),\ \forall n \in \underline{N}\ [R(n) \rightarrow R(n+1)]}{\forall m \in \underline{N}\ R(m)}$$

Like Dominos…

Example Induction Proof

Let’s prove:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Proof by Induction

Statements in green form a template for inductive proofs:

Proof: (by induction on *n*)

The induction hypothesis:

$$P(n) ::= \; 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Example Induction Proof

Base Case ($n = 0$):

$$\underbrace{1 + r + r^2 + \cdots + r^0}_{1} \stackrel{?}{=} \frac{r^{0+1} - 1}{r - 1}$$

$$= \frac{r - 1}{r - 1} = 1$$

Wait: divide by zero bug!

This is only true for $r \neq 1$

An Example Proof

Revised Theorem:

$$\forall r \neq 1 \quad 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Revised Induction Hypothesis:

$$P(n) ::= \forall r \neq 1 \; 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

An Example Proof

Induction Step: Assume $P(n)$ for $n \geq 0$ to prove $P(n+1)$:

$$\forall r \neq 1 \;\; 1 + r + r^2 + \cdots + r^{n+1} = \frac{r^{(n+1)+1} - 1}{r - 1}$$

An Example Proof

Have $P(n)$ by assumption:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Adding r^{n+1} to both sides:

$$1 + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1}$$

$$= \frac{r^{n+1} - 1 + r^{n+1}(r - 1)}{r - 1}$$

An Example Proof

Continued…

$$1+\cdots+r^{n}+r^{n+1} = \frac{r^{n+1}-1+r\cdot r^{n+1}+-r^{n+1}}{r-1}$$

$$= \frac{r^{(n+1)+1}-1}{r-1}$$

Which is just $P(n+1)$

Therefore theorem is true
by induction. QED.

An Aside: Ellipses

Ellipses (…) mean that the reader is supposed to *infer* a pattern.

- Can lead to confusion
- Summation notation gives more precision, for example:

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i$$

Problems

Class Problem 1

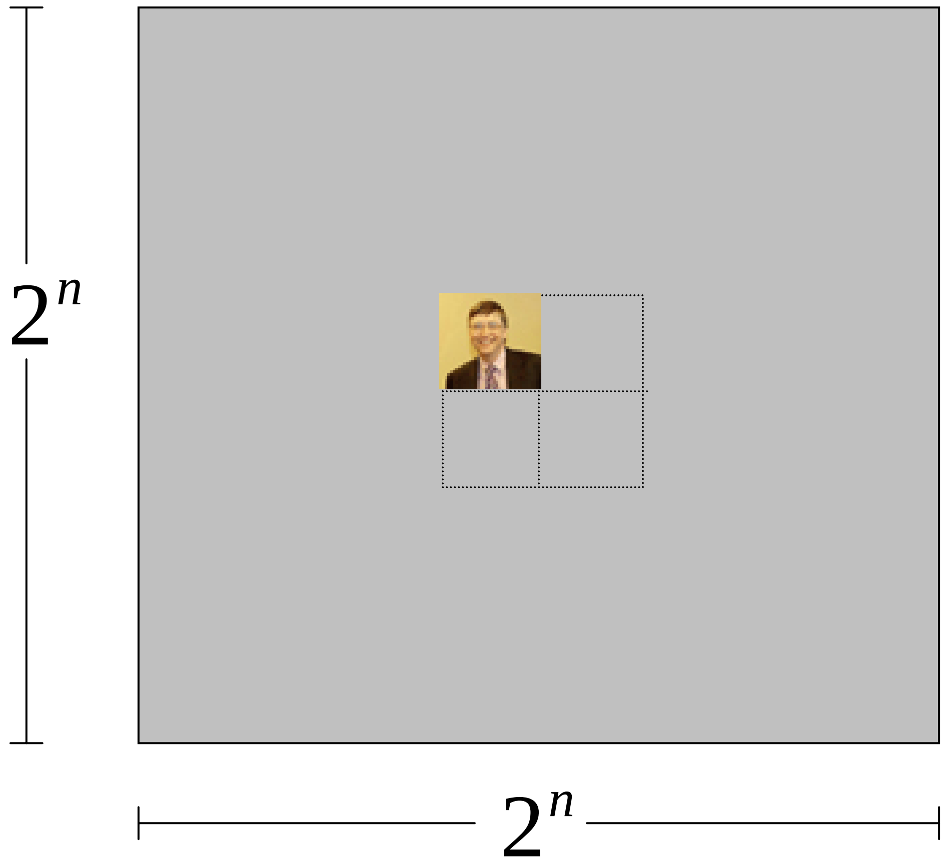
The MIT Stata Center

The Stata Center Plaza

The Gehry/Gates Plaza

Goal: tile the squares, except one in the middle for Bill.

Photo courtesy of Ricardo Stuckert/ABr.



The Gehry/Gates Plaza

Gehry specifies L-shaped tiles covering three squares:

For example, for 8 x 8 plaza might tile for Bill this way:

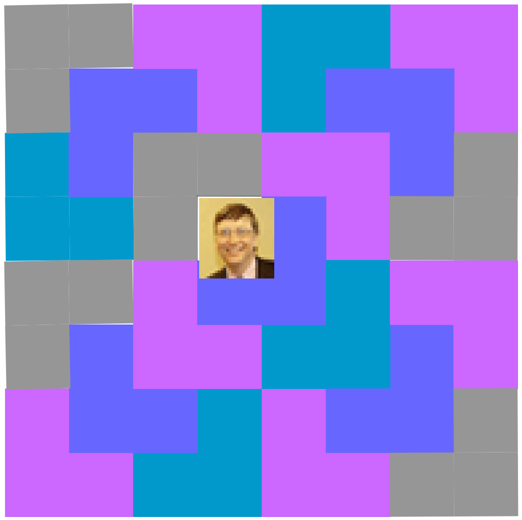


Photo courtesy of Ricardo Stuckert/ABr.

The Gehry/Gates Plaza

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill and Frank happy.

Proof: (by induction on n)

$P(n)$ ::= can tile $2^n \times 2^n$ with Bill in middle.

Base case: (n=0)

(no tiles needed)

Photo courtesy of Ricardo Stuckert/ABr.

The Gehry/Gates Plaza

Induction step: assume can tile $2^n \times 2^n$, prove can handle $2^{n+1} \times 2^{n+1}$.

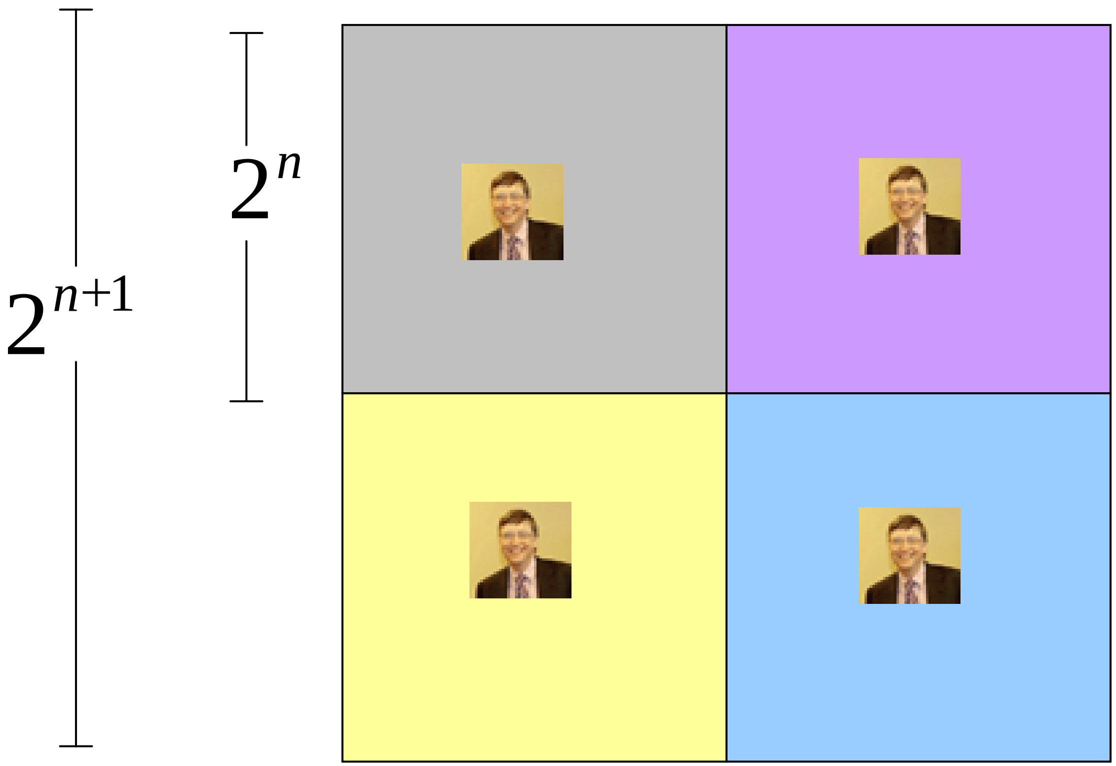


Photo courtesy of Ricardo Stuckert/ABr.

The Gehry/Gates Plaza

Now what?

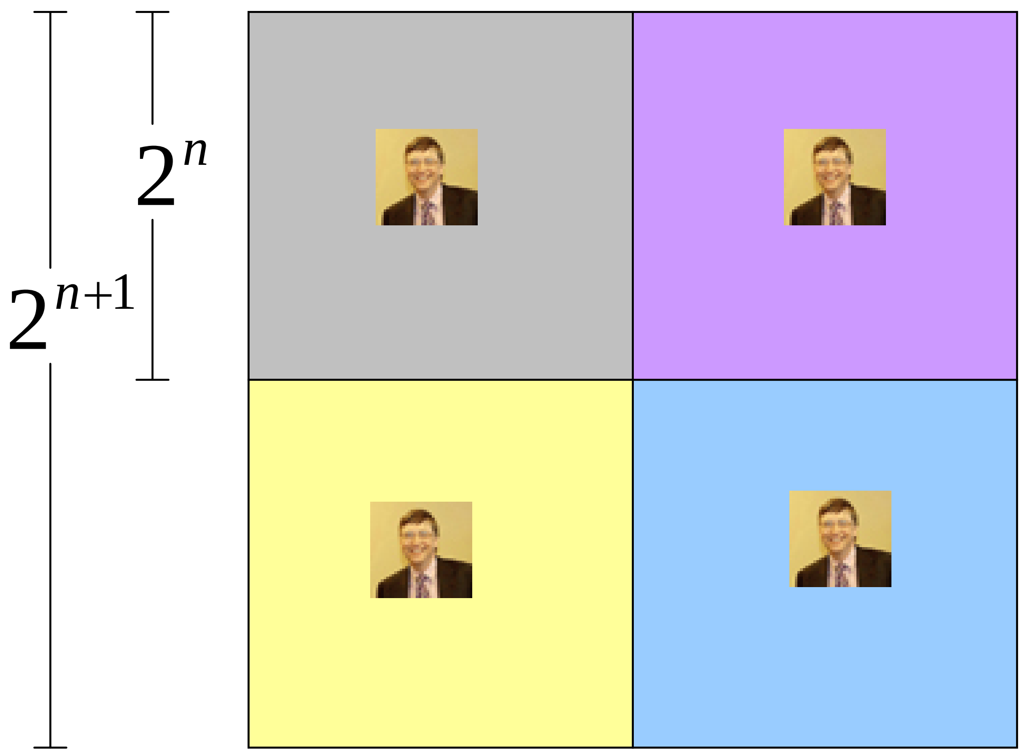


Photo courtesy of Ricardo Stuckert/ABr.

The Gehry/Gates Plaza

The fix:

Prove that we can always find a tiling with Bill in the corner.

The Gehry/Gates Plaza

Note: Once have Bill in corner, can get Bill in middle:

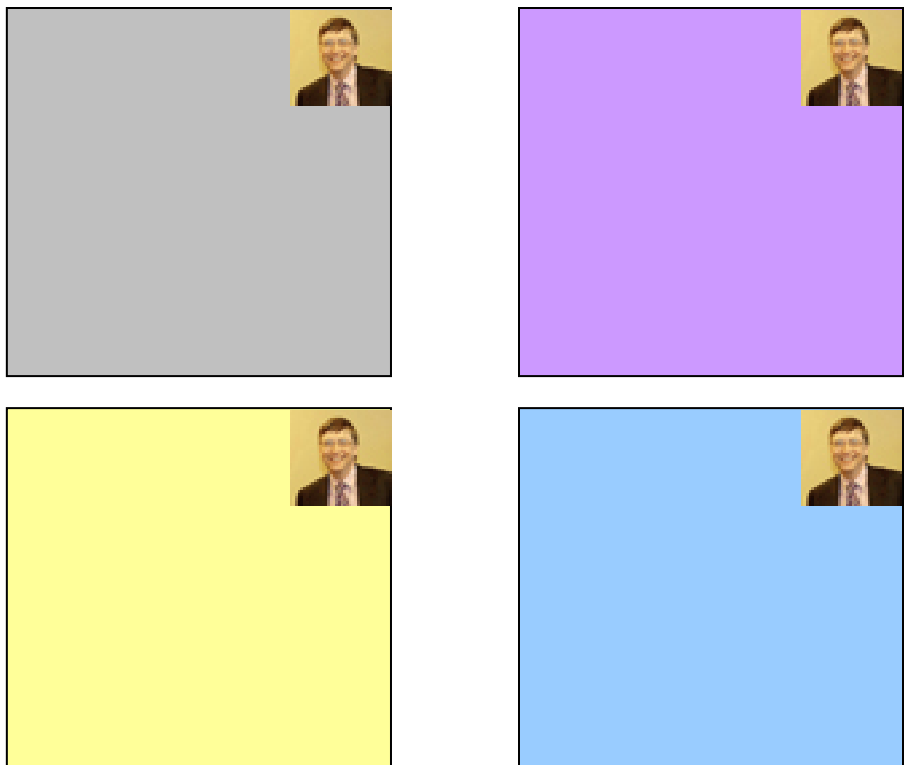


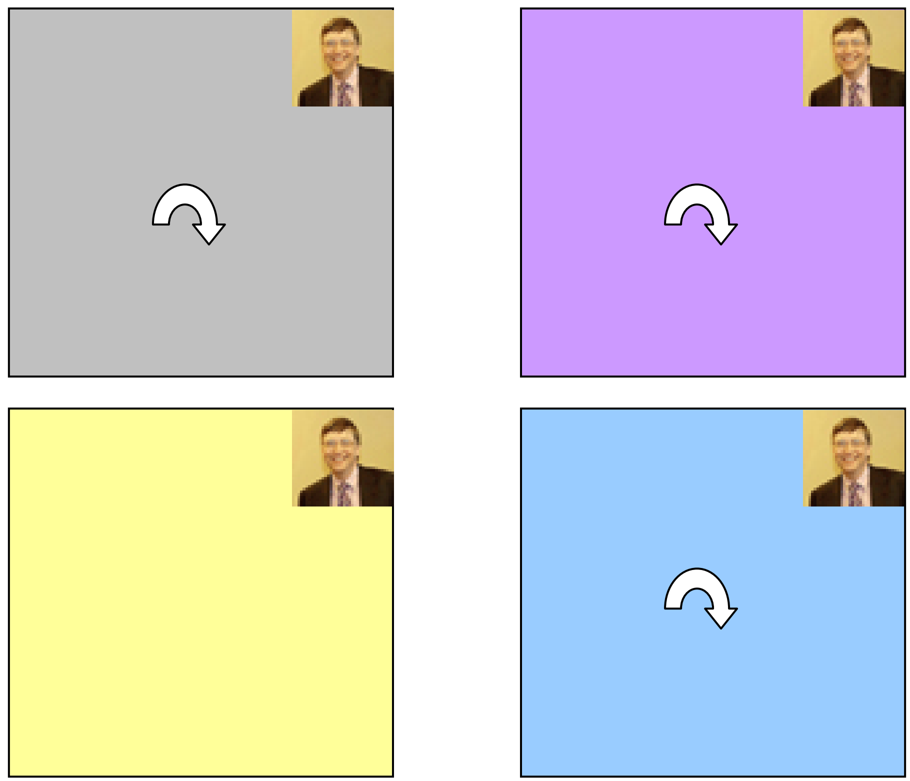
Photo courtesy of Ricardo Stuckert/ABr.

The Gehry/Gates Plaza

Method:

Rotate the squares as indicated.

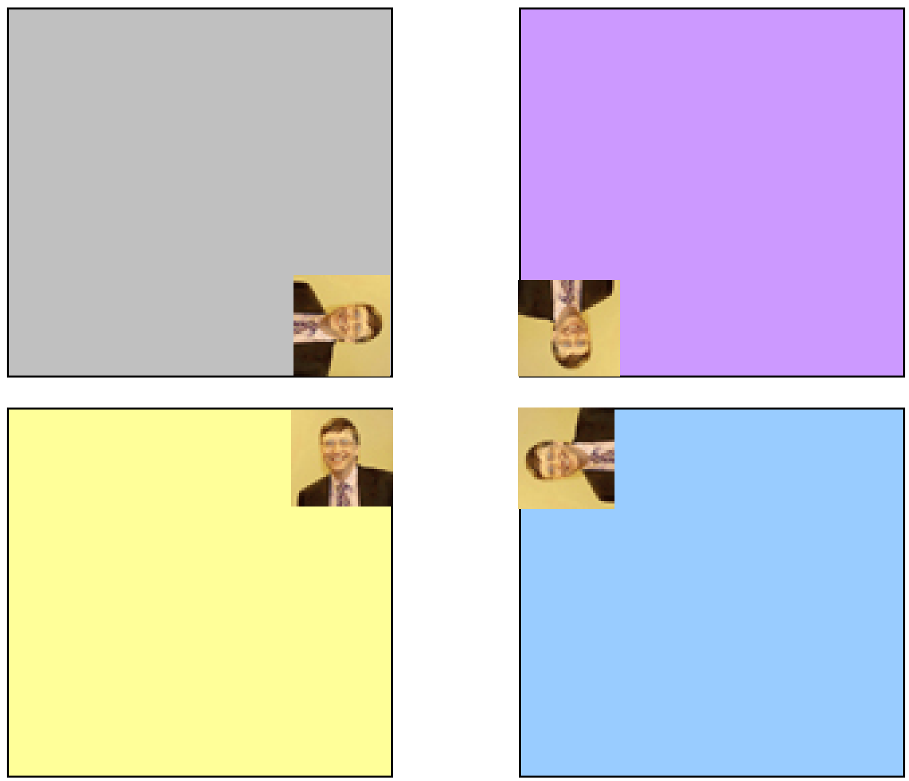
Photo courtesy of Ricardo Stuckert/ABr.



The Gehry/Gates Plaza

Method: after rotation have:

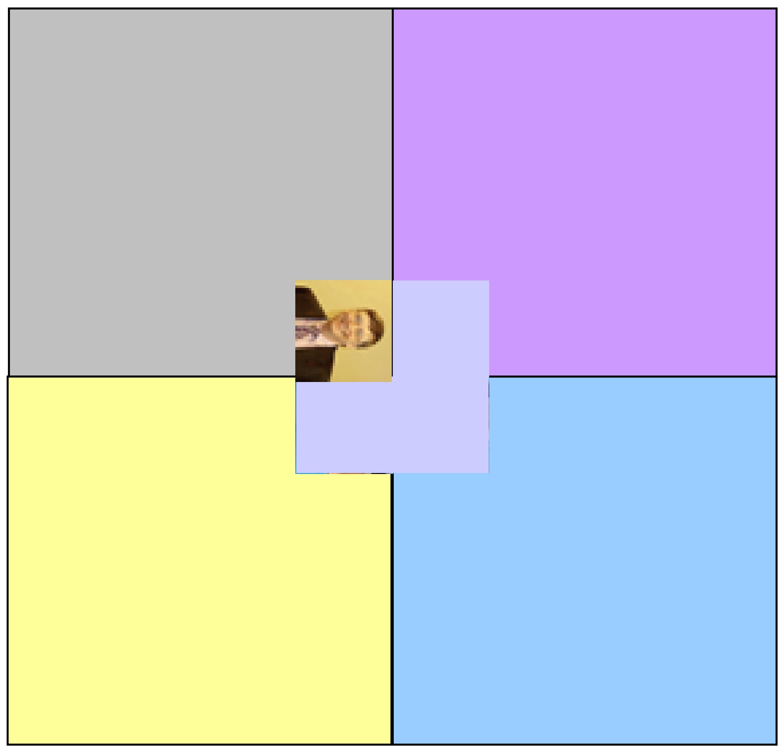
Photo courtesy of Ricardo Stuckert/ABr.



The Gehry/Gates Plaza

Method: Now group the 4 squares together, and insert a tile.

Photo courtesy of Ricardo Stuckert/ABr.



Done!

Bill in middle

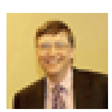
The Gehry/Gates Plaza

Theorem: For any $2^n \times 2^n$ plaza, we can put Bill in the corner.

Proof: (by induction on n)

$P(n)$ ::= can tile $2^n \times 2^n$ with Bill in corner

Base case: (n=0)



(no tiles needed)

Photo courtesy of Ricardo Stuckert/ABr.

The Gehry/Gates Plaza

Induction step:

Assume we can get Bill in corner of $2^n \times 2^n$.

Prove we can get Bill in corner of $2^{n+1} \times 2^{n+1}$.

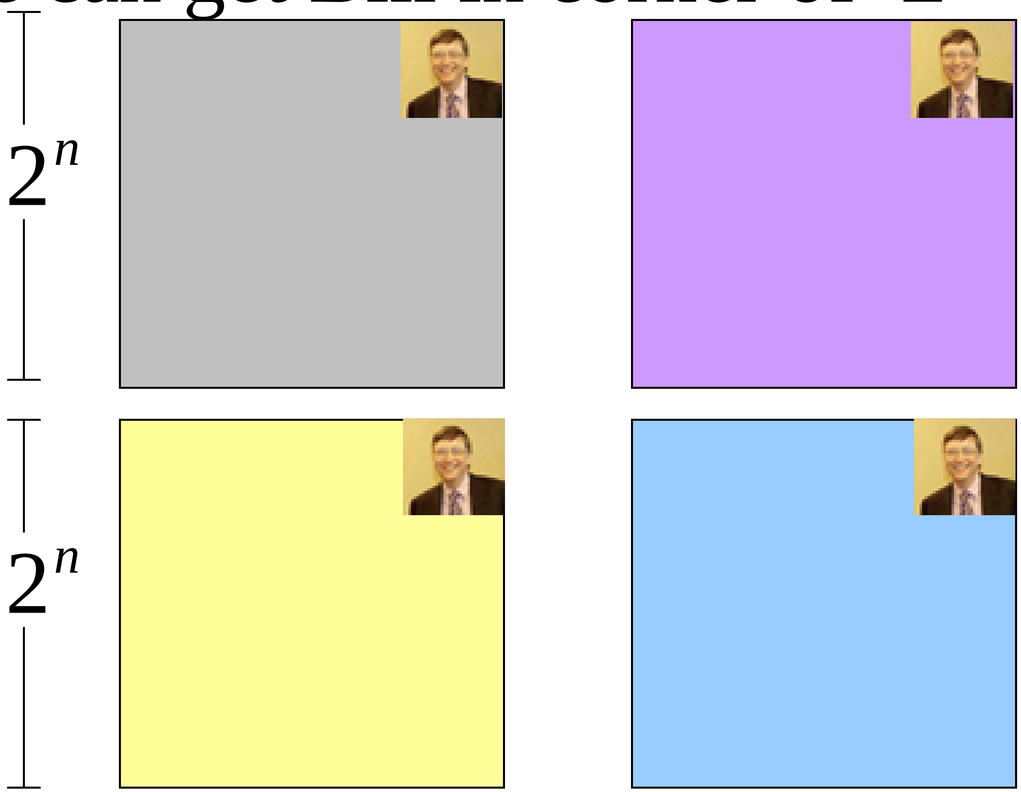
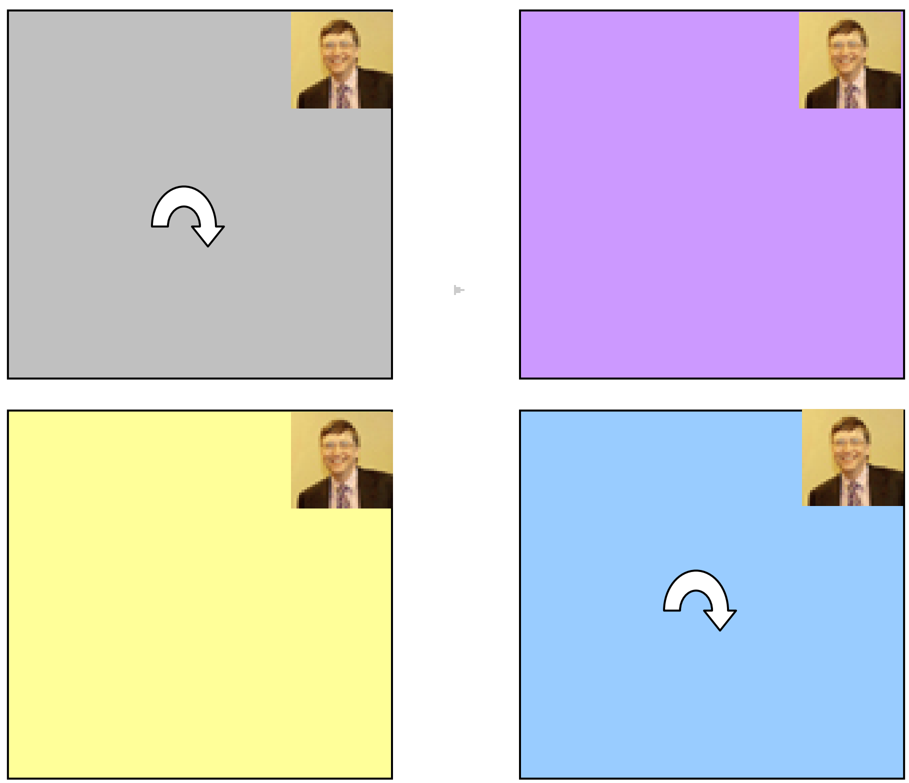


Photo courtesy of Ricardo Stuckert/ABr.

The Gehry/Gates Plaza

Method: Rotate the squares as indicated.

Photo courtesy of Ricardo Stuckert/ABr.

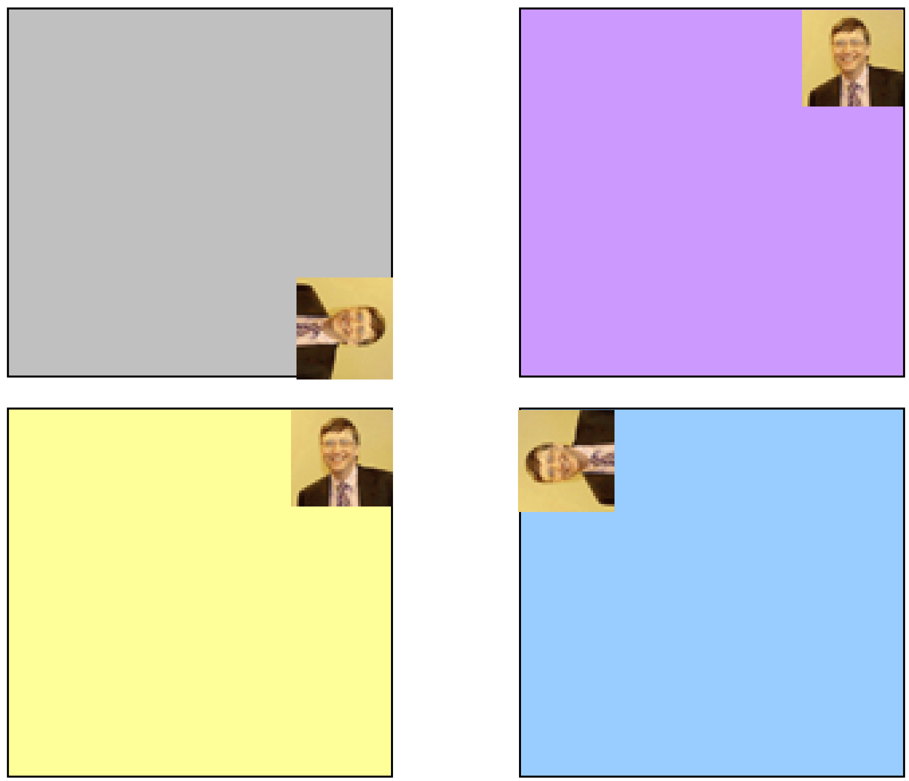


The Gehry/Gates Plaza

Method: Rotate the squares as indicated.

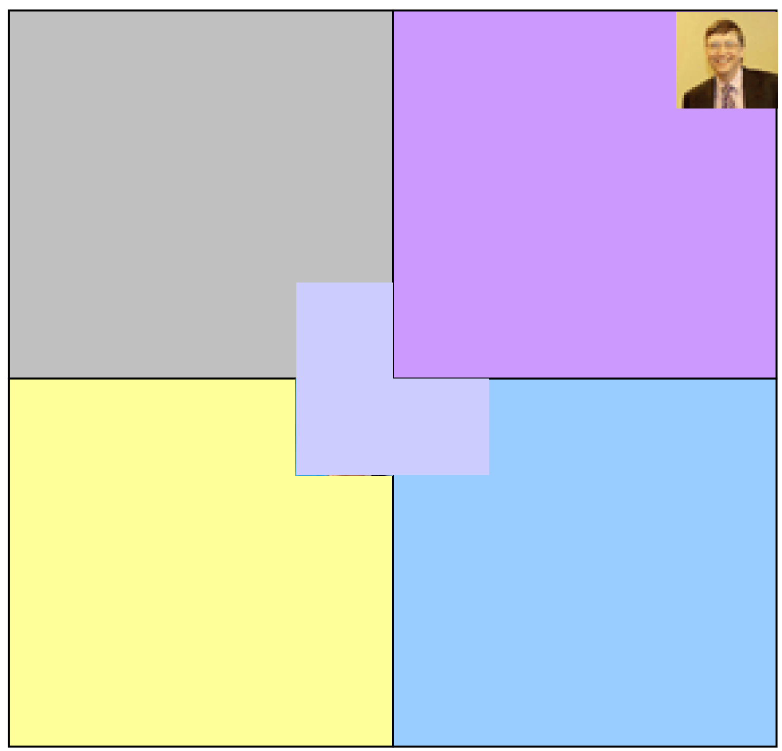
after rotation have:

Photo courtesy of Ricardo Stuckert/ABr.



The Gehry/Gates Plaza

Method: Now group the squares together, and fill the center with a tile.



Done!

Photo courtesy of Ricardo Stuckert/ABr.

Ingenious Induction Hypotheses

Note 1: To prove

"Bill in middle,"

we proved *something else:* "Bill in corner."

Ingenious Induction Hypotheses

Note 2: Other times it helps to choose a *stronger hypothesis* than the desired result.

Inductive (Recursive) Procedures

Note 3: The induction proof of “Bill in corner” implicitly defines a recursive procedure for constructing a $2^{n+1} \times 2^{n+1}$ corner tiling from a $2^n \times 2^n$ corner tiling.

Problems

Class Problem 2

A False Proof

Theorem: All horses are the same color.

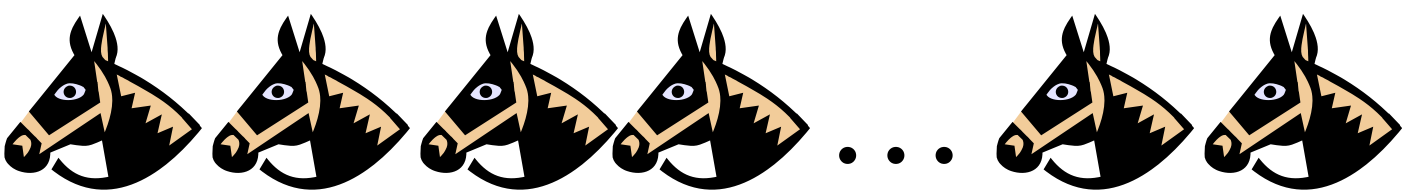
Proof: (by induction on *n*)

Induction hypothesis:

$P(n)$ ::= any set of *n* horses have the same color

Base case (n=0):

No horses so *vacuously* true!

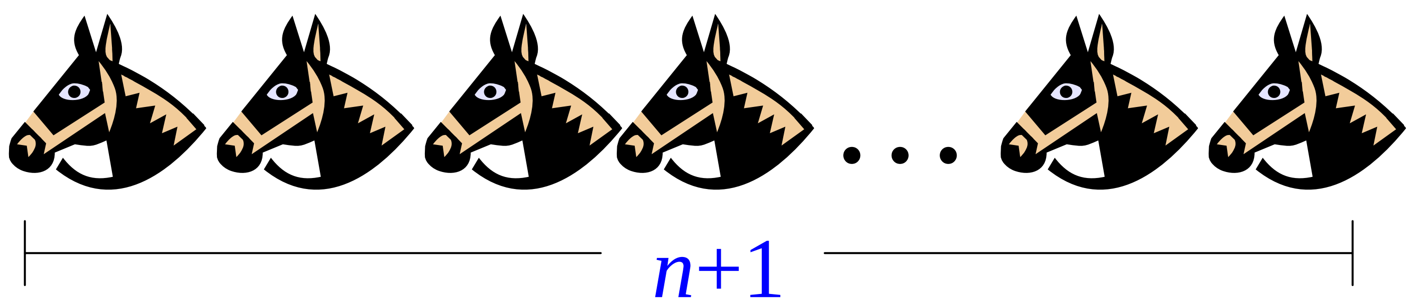


A False Proof

(Inductive case)

Assume any n horses have the same color.

Prove that any $n+1$ horses have the same color.



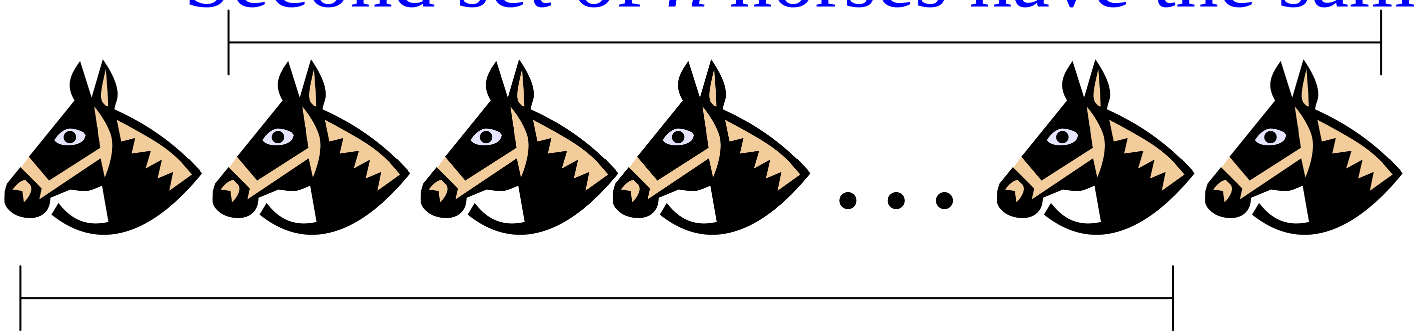
A False Proof

(Inductive case)

Assume any n horses have the same color.

Prove that any $n+1$ horses have the same color.

Second set of n horses have the same color



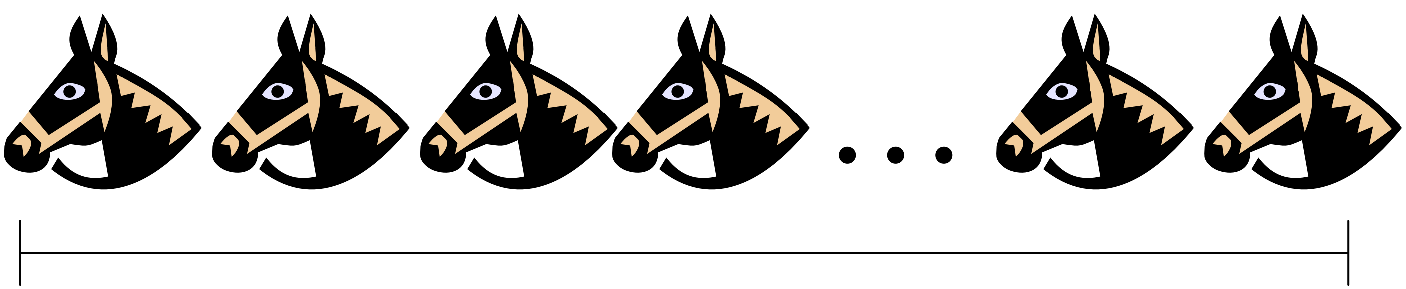
First set of n horses have the same color

A False Proof

(Inductive case)

Assume any n horses have the same color.

Prove that any $n+1$ horses have the same color.



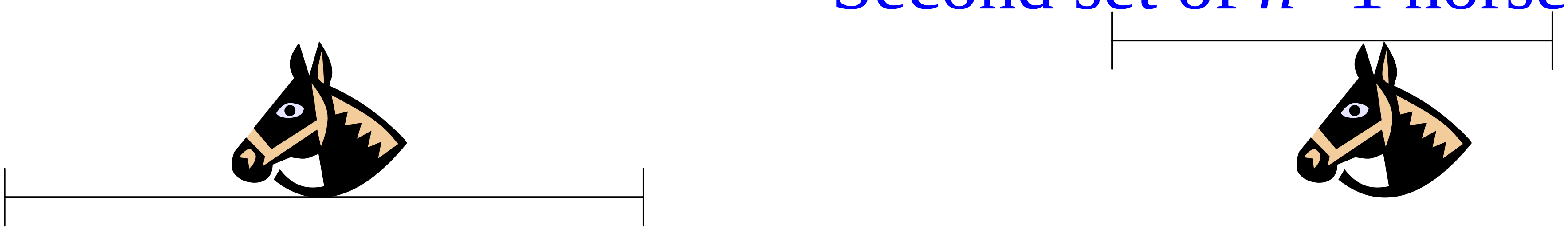
Therefore the set of $n+1$ have the same color!

A False Proof

What is wrong? $n = 1$

Proof that $P(n) \to P(n+1)$ is false if $n = 1$, because the two horse groups *do not overlap*.

Second set of n=1 horses



First set of n=1 horses

A False Proof

Proof that $P(n) \rightarrow P(n+1)$ is false if $n = 1$, because the two horse groups *do not overlap*.

(But proof works for all $n \neq 1$)