Mathematics for Computer Science

MIT 6.042J/18.062J

Predicate Logic
Quantifiers ∀, ∃

Predicates

Predicates are

Propositions with variables

Example:

$P(x,y)$ ::= $x+2=y$

"is defined to be"

Predicates

$$P(x,y) \ ::= [x+2=y]$$

$x = 1$ and $y = 3$: $P(1,3)$ is true

$x = 1$ and $y = 4$: $P(1,4)$ is false

$\neg P(1,4)$ is true

Quantifiers

$\forall x$ For ALL x

$\exists y$ There EXISTS some y

Quantifiers

x, y range over ***Domain of Discourse***

$$\forall x \exists y \ x < y$$

Domain	Truth value
integers $\mathbb{Z}$	True
positive integers $\mathbb{Z}^+$	True
negative integers $\mathbb{Z}^-$	False
negative reals $\mathbb{R}^-$	True

Team Problems

Problems

1 & 2

Math vs. English

Poet:

"All that glitters is not gold."

(glitters = G; gold = Au)

$$\forall x\, G(x) \longrightarrow \neg Au(x)$$

No!: gold glitters like gold

Math vs. English

Poet:

"All that glitters is not ^necessarily gold."

(glitters = G; gold = Au)

$$\neg[\forall x\, G(x) \longrightarrow Au(x)]$$

(Poetic license)

Math vs. English

Poet: "There is season for every purpose under heaven"

$\exists s \in$ season $\forall p \in$ purpose
s is the season for p

No!

Math vs. English

Poet: "There is season for every purpose under heaven"

$\forall p \in$ purpose $\exists s \in$ season
s is the season for p

(Poetic license again.)

Propositional Validity

$$(A \rightarrow B) \vee (B \rightarrow A)$$

True ***no matter what*** the truth values of A and B are

Predicate Calculus Validity

$$\forall z\, [Q(z) \wedge P(z)] \rightarrow [\forall x Q(x) \wedge \forall y P(y)]$$

True ***no matter what***

- the Domain is,
- the predicates *are*.

Not Valid

$\forall z\,[Q(z) \lor P(z)] \rightarrow [\forall x Q(x) \lor \forall y P(y)]$

Proof: Give *countermodel*, where $\forall z\,[Q(z) \lor P(z)]$ is true, but $\forall x Q(x) \lor \forall y P(y)$ is false.

Namely, let domain ::= $\{e, \pi\}$,
$Q(z)$ ::= $[z = e]$,
$P(z)$ ::= $[z = \pi]$.

Validities

$\forall z\,[Q(z) \land P(z)] \rightarrow [\forall x Q(x) \land \forall y P(y)]$

Proof strategy: We assume
$\forall z\,[Q(z) \land P(z)]$
to prove
$\forall x Q(x) \land \forall y P(y)$.

Predicate Inference Rule

$$\frac{Q \rightarrow P(c)}{Q \rightarrow \forall x.P(x)}$$

(providing c does not occur in Q)

Universal Generalization (UG)

Validities

$\forall z\,[Q(z) \land P(z)] \rightarrow [\forall x Q(x) \land \forall y P(y)]$

Proof: Assume $\forall z\,[Q(z) \land P(z)]$.
So $Q(z) \land P(z)$ holds for all z in the domain.
Now let c be some domain element. So $Q(c) \land P(c)$ holds, and therefore $Q(c)$ by itself holds.
But c could have been any element of the domain.
So we conclude $\forall x Q(x)$. (by *UG*)
We conclude $\forall y P(y)$ similarly. Therefore,
$\forall x Q(x) \land \forall y P(y)$ QED.

More Validities

$\forall x\,[P(x) \lor A] \leftrightarrow [\forall x\, P(x)] \lor A$

(providing x not in A)

$[\neg \forall x\, P(x)] \leftrightarrow [\exists x\, \neg P(x)]$

Team Problems

Problems

3 & 4

