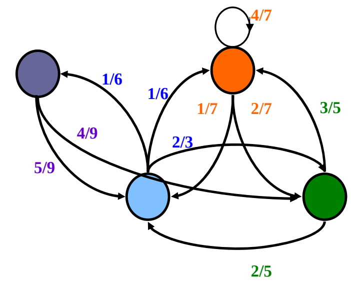
Mathematics for Computer Science

MIT 6.042J/18.062J

Random Walks

 lec 15w.1

Graph with probable transitions



 lec 15w.2

Graph with probable transitions

Questions

- Pr{blue reaches orange *before* green}□
- Pr{blue *ever* reaches orange}
- E[#steps blue to orange]
- Average fraction of time at blue

 lec 15w.3

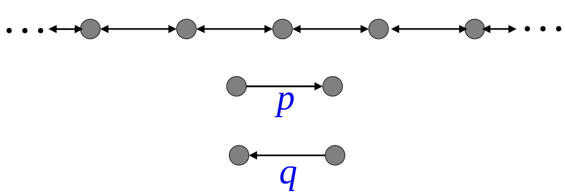
Random Walks

Applications

- Finance – Stocks, options
- Algorithms – web search, clustering
- Physics – Brownian Motion

 lec 15w.4

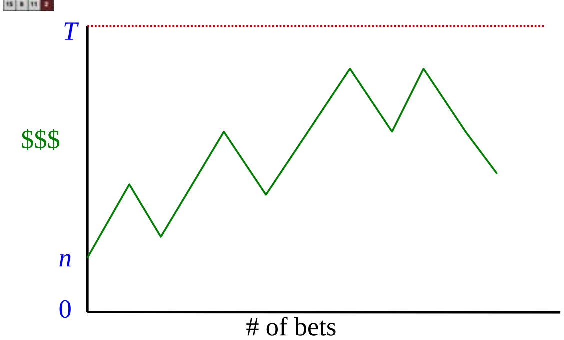
1-Dimensional Walk



Gambler's Ruin

 lec 15w.5

Gambler's Ruin



 lec 15w.6

Gambler's Ruin

Parameters:

n ::= initial capital (stake)

T ::= gambler's Target

p ::= Pr{win \$1 bet}

q ::= $1 - p$

m ::= intended profit = $T - n$

Gambler's Ruin

Three general cases:

- Biased against $p < 1/2$
- Biased in favor $p > 1/2$
- Unbiased (Fair) $p = 1/2$

Fair Case: $p = q = 1/2$

Let w ::= Pr{reach Target}

$$E[\$\$] = w\cdot(T-n) + (1-w)\cdot(-n)$$
$$= wT - n$$

But game is *fair*, so E[\$\$ won] = 0

$$w = \frac{n}{T}$$

Fair Case: $p = q = 1/2$

Let w ::= Pr{reach Target}

$$w = \frac{n}{T}$$

Fair Case

Consequences

n=500, T=600

Pr{win \$100} = 500/600 ≈ 0.83

n=1,000,000, T=1,000,100

Pr{win \$100} ≈ 0.9999

Biased Against: $p < 1/2 < q$

Betting red in US roulette

$$p = 18/38 = 9/19 < 1/2$$

Biased Against: $p < 1/2 < q$

Astonishing Fact!

Pr{win \$100 starting with \$500}

< 1/37,000 !

(was 5/6 in the unbiased case.)

Biased Against: $p < 1/2 < q$

More amazing still!

Pr{win \$100 starting with \$1M}

< 1/37,000

Pr{win \$100 starting w/ any \$*n* stake}

< 1/37,000

Winning in the Unfair Case

Team Problem: for $p < q$,

$$w_n \le \frac{(q/p)^n}{(q/p)^T} = \left(\frac{p}{q}\right)^m$$

where $m ::= T - n$ = intended profit

Winning in the Unfair Case

for $p < q$:

$$\left(\frac{p}{q}\right)^m$$

is exponentially decreasing in m,

the intended profit.

Losing in Roulette

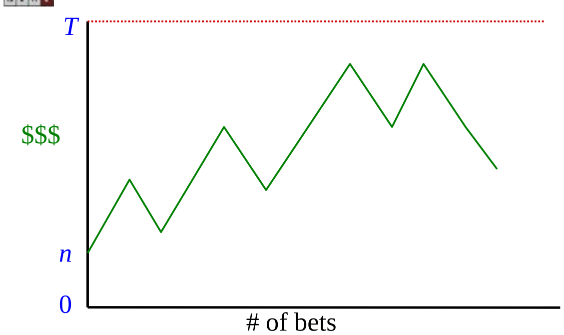
$p = 18/38$, $q = 20/38$

$$\Pr\{\text{win } \$100\} = \left(\frac{18/38}{20/38}\right)^{100} = \left(\frac{9}{10}\right)^{100} < \frac{1}{37,648}$$

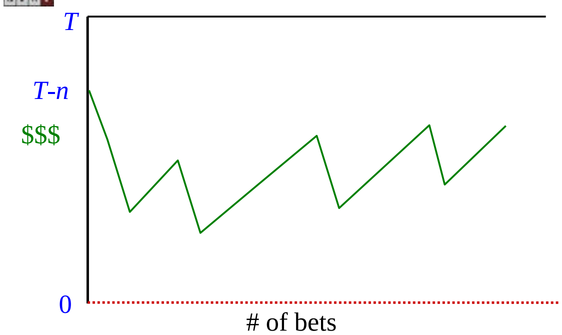
Losing in Roulette

$$\Pr\{\text{win } \$200\} = (\Pr\{\text{win } \$100\})^2 = \left(\frac{1}{37,648}\right)^2 < \frac{1}{70,000,000}$$

Gambler's Ruin



Gambler's Ruin



Fair Case

$pr\{$lose starting with $\$n\}$

$$= pr\{\text{win starting with } \$(T-n)\}$$

$$= \frac{T-n}{T}$$

Fair Case for $T = \infty$

$$Pr\{\text{lose starting with } n \mid T = \infty\}$$

$$\geq Pr\{\text{lose starting with } n \mid T < \infty\}$$

$$= \frac{T-n}{T} \to \infty \qquad \text{as } T \to \infty$$

So if the gambler keeps betting, he is sure to go broke.

Return to the origin.

If you start at the origin and move left or right with equal probability, and keep moving in this way,

Pr{return to origin} = 1

How Many Bets?

What is the expected number of bets for the game to end?

– either by winning $\$(T\text{-}n)$ or by going broke (losing $\$n$).

How Many Bets? Fair Case

E[# bets] = $n(T-n)$ =
(initial stake)·(intended profit)
proof by solving linear recurrence:

$$e_n = p(1 + e_{n+1}) + q(1 + e_{n-1})$$

Fair Case for $T = \infty$

Likewise,

E[#bets for $T = \infty$]
$\geq$ E[#bets for $T < \infty$]
$= n(T\text{-}n) \to \infty$ (as $T \to \infty$)

So the expected #bets to go broke is infinite!

Team Problems

Problems 1—3