

Solutions to In-Class Problems Week 9, Wed.

Problem 1. A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)
- 2 characters – letters or numbers (big shot plate)

Let L be the set of all possible license plates.

(a) Express L in terms of

$$\begin{aligned}\mathcal{A} &= \{A, B, C, \dots, Z\} \\ \mathcal{D} &= \{0, 1, 2, \dots, 9\}\end{aligned}$$

using unions (\cup) and set products (\times).

Solution.

$$L = (A^3 \times D^3) \cup A^5 \cup (A \cup D)^2$$



(b) Compute $|L|$, the number of different license plates, using the sum and product rules.

Solution.

$$\begin{aligned}|L| &= |(A^3 \times D^3) \cup A^5 \cup (A \cup D)^2| \\ &= |(A^3 \times D^3)| + |A^5| + |(A \cup D)^2| && \text{Sum Rule} \\ &= |A|^3 \cdot |D|^3 + |A|^5 && |A \cup D|^2 \text{Product Rule} \\ &= |A|^3 \cdot |D|^3 + |A|^5 && (|A| + |D|)^2 \text{Sum Rule} \\ &= 26^3 \cdot 10^3 + 26^5 + 36^2 = 29458672\end{aligned}$$



Problem 2. Let p be a prime and k a positive integer.

(a) How many positive integers less than p^k are divisible by p ?

Solution. Every p th integer is divisible by p , namely $p, 2p, 3p, \dots, (p^{k-1} - 1)p$, so the answer is $p^{k-1} - 1$. ■

(b) What is the value, $\phi(p^k)$, of the Euler function at p^k ?

Solution. There are $p^k - 1$ positive integers less than p^k , of which $p^{k-1} - 1$ are divisible by p by part (a), and $\phi(p^k)$ are relatively prime to p^k by definition of Euler's function. But a number is relatively prime to p^k iff it is not divisible by p , so

$$(p^{k-1} - 1) + \phi(p^k) = p^k - 1$$

by the Sum Rule. Therefore,

$$\phi(p^k) = p^k - p^{k-1}. \quad \blacksquare$$

Problem 3. For each part below, describe a bijection between the two sets mentioned. The existence of such a bijection proves that the two sets are the same size.

A good approach is to describe an element of the first set using variables and then describe the corresponding element of the second set in terms of those variables. For example, we might describe a bijection from ways of selecting a dozen doughnuts from five varieties to a 16-bit string with four 1's as follows:

Map a dozen doughnuts consisting of:

c chocolate, l lemon-filled, s sugar, g glazed, and p plain

to the sequence:

$$\underbrace{0 \dots 0}_c \quad 1 \quad \underbrace{0 \dots 0}_l \quad 1 \quad \underbrace{0 \dots 0}_s \quad 1 \quad \underbrace{0 \dots 0}_g \quad 1 \quad \underbrace{0 \dots 0}_p$$

Everyone in your group should write out complete answers—you'll all benefit from the practice!

(a) Describe a bijection between the set of 30-bit sequences with 10 zeros and 20 ones and paths from $(0, 0)$ to $(10, 20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate).

Solution. Map the 30-bit sequence $b_1 b_2 \dots b_{30}$ to a path where the i -th step is right if $b_i = 0$ and up if $b_i = 1$. ■

(b) Find a bijection between the set of n -bit sequences and the set of all subsets of $\{x_1, x_2, \dots, x_n\}$.

Solution. Map the n -bit sequence $b_1b_2 \dots b_n$ to a subset that contains x_i if and only if $b_i = 1$. ■

(c) Mr. and Mrs. Grumperson have collected 13 identical pieces of coal as Christmas presents for their beloved children, Lucy and Spud. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the two children and the set of 14-bit sequences with exactly 1 one.

Solution. Map a distribution in which Lucy get l pieces and Spud gets s pieces to a 14-bit sequence with l zeros, a one, and then s zeros. ■

(d) On Christmas Eve, Mr. and Mrs. Grumperson remember that they have a third child, little Bottlecap, locked in the attic. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the three children and the set of 15-bit sequences with exactly 2 ones.

Solution. Map a distribution in which Lucy gets l pieces, Spud gets s pieces, and Bottlecap gets b pieces to a 15-bit sequence with l zeros, a one, s zeros, a one, and b zeros. ■

(e) On reflection, Mr. and Mrs. Grumperson decide that each of their three children should receive *at least two* pieces of coal for Christmas. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the three Grumperson children given this constraint and the set of 9-bit sequences with exactly 2 ones.

Solution. Map a distribution in which Lucy gets $l \geq 2$ pieces, Spud gets $s \geq 2$ pieces, and Bottlecap gets $b \geq 2$ pieces to a 9-bit sequence with exactly $l - 2$ zeros, a one, $s - 2$ zeros, a one, and $b - 2$ zeros. ■

(f) Describe a bijection between the set of 110-bit sequences with exactly 10 ones and solutions over the natural numbers to the equation:

$$x_1 + x_2 + \dots + x_{10} \leq 100$$

Solution. Let x_1 be the number of zeros before the first 1, x_2 , be the number of zeros between the first and second 1, etc. Note that zeros after the tenth 1 do not contribute to the value of any of the variables x_1, \dots, x_{10} ; this allows us to count solutions to the inequality (≤ 100) rather than the equality ($= 100$). ■

(g) Describe a bijection between solutions to the inequality in the preceding problem part and sequences $(y_1, y_2, \dots, y_{10})$ such that:

$$0 \leq y_1 \leq y_2 \leq \dots \leq y_{10} \leq 100$$

Solution. Let $y_i = x_1 + \cdots + x_i$ for each i from 1 to 10. ■

Problem 4. A *numbered tree* is a tree whose vertex set is $\{1, 2, \dots, n\}$ for some $n \geq 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers from 1 to n obtained by the following recursive process:

If $n = 2$, stop—the code is the empty sequence. Otherwise, write down the *father* of the largest leaf¹, delete this *leaf*, and continue the process on the resulting smaller tree.

For example, the codes of a couple of numbered trees are shown in the Figure 1.

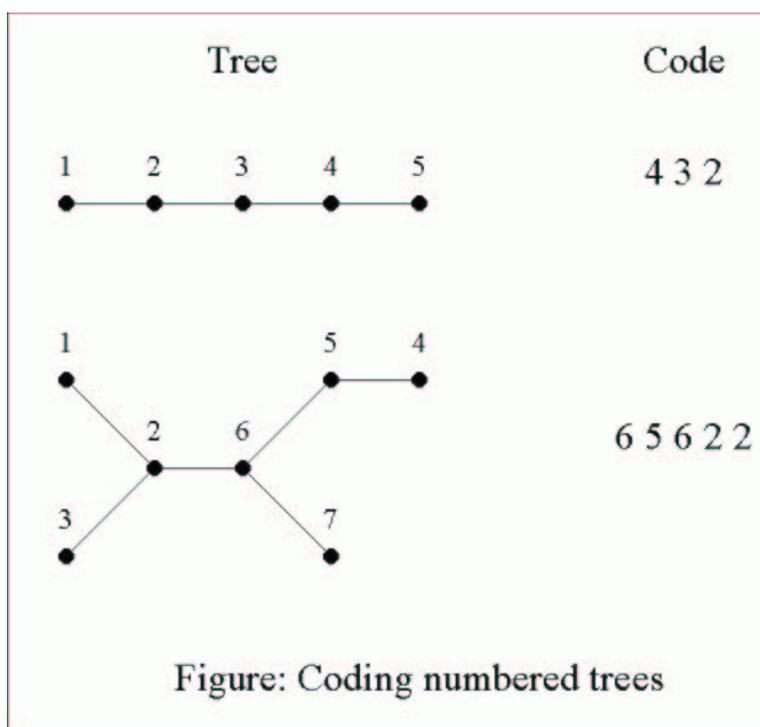


Figure 1:

(a) Describe a procedure for reconstructing a numbered tree from its code.

Solution. The key observation is that, given a code of length $n - 2$, the numbers between 1 and n which *do not appear* in the code must be leaves of the tree. Hence, the largest missing number is a leaf attached to the first number of the code. The rest of the tree can now be reconstructed by deleting the first number in the code, henceforth ignoring the largest leaf, and proceeding

¹The necessarily unique node adjacent to a leaf is called its *father*

recursively on the rest of the code. (We're using the obvious fact that what's left after deleting a leaf from a tree is another tree.)

More precisely, the reconstruction procedure applies to any finite tree whose vertex set is totally ordered. The procedure takes *two* parameters: the vertex set, V , and a length $|V| - 2$ "code" sequence, S , of elements in V . If l is the largest element in V which does not appear in S , and f is the first element of S , then the reconstructed tree is obtained by adding edge (l, f) to the tree reconstructed by calling the procedure recursively with first argument $V - \{l\}$ and second argument equal to the code obtained by erasing the initial f from S . The procedure terminates when $|V| = 2$, returning the edge between the two numbers in V .

To justify the key observation, note that any vertex that gets deleted by the process and was not a leaf to begin with, must have been the father of a previously deleted leaf, which means it would appear in the code. So the missing integers must have been leaves to begin with or must be one of the two undeleted vertices left when the coding process terminates. But by the end of the process the two remaining vertices are leaves, and if they weren't leaves to begin with, they must have become leaves by having their sons deleted, which means they would not have been missing from the code. So the two vertices remaining at the end must also have been leaves of the original tree.

■

(b) How many numbered trees with n vertices are there? Justify your answer assuming the result of the previous problem part.

Solution. There are exactly as many n -vertex numbered trees as the number of possible code words, that is, the number of length $n - 2$ sequences integers between 1 and n . So there are n^{n-2} numbered trees.

The reason is that the map from trees to codes is a bijection. To see this, note that the tree reconstruction procedure finds *the only possible tree* with that code. So there can't be two trees with the same code, i.e., the map from a tree to its code is an injection. But since the reconstruction procedure finds a tree for every possible codeword, the map from trees to codes is also a surjection.

■