

In-Class Problems Week 9, Wed.

Problem 1. A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)
- 2 characters – letters or numbers (big shot plate)

Let L be the set of all possible license plates.

(a) Express L in terms of

$$\mathcal{A} = \{A, B, C, \dots, Z\}$$

$$\mathcal{D} = \{0, 1, 2, \dots, 9\}$$

using unions (\cup) and set products (\times).

(b) Compute $|L|$, the number of different license plates, using the sum and product rules.

Problem 2. Let p be a prime and k a positive integer.

- (a) How many positive integers less than p^k are divisible by p ?
- (b) What is the value, $\phi(p^k)$, of the Euler function at p^k ?

Problem 3. For each part below, describe a bijection between the two sets mentioned. The existence of such a bijection proves that the two sets are the same size.

A good approach is to describe an element of the first set using variables and then describe the corresponding element of the second set in terms of those variables. For example, we might describe a bijection from ways of selecting a dozen doughnuts from five varieties to a 16-bit string with four 1's as follows:

Map a dozen doughnuts consisting of:

c chocolate, l lemon-filled, s sugar, g glazed, and p plain

to the sequence:

$$\underbrace{0\dots 0}_c \quad 1 \quad \underbrace{0\dots 0}_l \quad 1 \quad \underbrace{0\dots 0}_s \quad 1 \quad \underbrace{0\dots 0}_g \quad 1 \quad \underbrace{0\dots 0}_p$$

Everyone in your group should write out complete answers— you'll all benefit from the practice!

(a) Describe a bijection between the set of 30-bit sequences with 10 zeros and 20 ones and paths from $(0, 0)$ to $(10, 20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate).

(b) Find a bijection between the set of n -bit sequences and the set of all subsets of $\{x_1, x_2, \dots, x_n\}$.

(c) Mr. and Mrs. Grumperson have collected 13 identical pieces of coal as Christmas presents for their beloved children, Lucy and Spud. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the two children and the set of 14-bit sequences with exactly 1 one.

(d) On Christmas Eve, Mr. and Mrs. Grumperson remember that they have a third child, little Bottlecap, locked in the attic. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the three children and the set of 15-bit sequences with exactly 2 ones.

(e) On reflection, Mr. and Mrs. Grumperson decide that each of their three children should receive *at least two* pieces of coal for Christmas. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the three Grumperson children given this constraint and the set of 9-bit sequences with exactly 2 ones.

(f) Describe a bijection between the set of 110-bit sequences with exactly 10 ones and solutions over the natural numbers to the equation:

$$x_1 + x_2 + \cdots + x_{10} \leq 100$$

(g) Describe a bijection between solutions to the inequality in the preceding problem part and sequences $(y_1, y_2, \dots, y_{10})$ such that:

$$0 \leq y_1 \leq y_2 \leq \dots \leq y_{10} \leq 100$$

Problem 4. A *numbered tree* is a tree whose vertex set is $\{1, 2, \dots, n\}$ for some $n \geq 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers from 1 to n obtained by the following recursive process:

If $n = 2$, stop—the code is the empty sequence. Otherwise, write down the *father* of the largest leaf¹, delete this *leaf*, and continue the process on the resulting smaller tree.

For example, the codes of a couple of numbered trees are shown in the Figure 1.

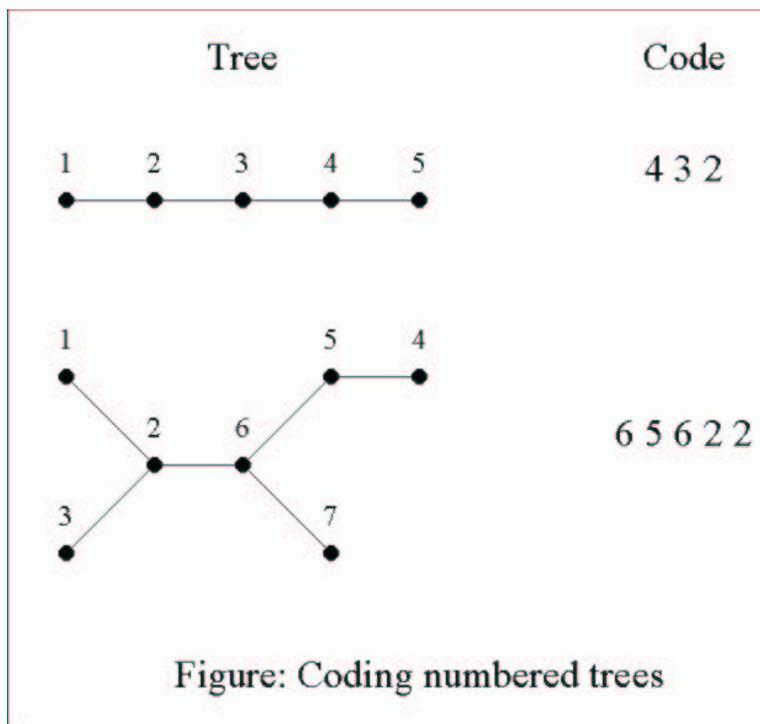


Figure 1:

- (a) Describe a procedure for reconstructing a numbered tree from its code.
- (b) How many numbered trees with n vertices are there? Justify your answer assuming the result of the previous problem part.

¹The necessarily unique node adjacent to a leaf is called its *father*