# Solutions to In-Class Problems Week 9, Mon.

**Problem 1.** Prove that asymptotic equality ($\sim$) is an equivalence relation.

**Solution.** reflexivity: $\lim_{x\to\infty} f(x)/f(x) = 1$, so $f \sim f$.

symmetry: Say $f \sim g$. Then $\lim_{x\to\infty} f(x)/g(x) = 1$. So $\lim_{x\to\infty} g(x)/f(x) = \lim_{x\to\infty} 1/(f(x)/g(x)) = 1/\lim_{x\to\infty} f(x)/g(x) = 1/1 = 1$, and therefore $g \sim f$.

transitivity: Say $f \sim g$ and $g \sim h$. So

$$\begin{aligned} 1 &= 1 \cdot 1 \\ &= [\lim_{x\to\infty} f(x)/g(x)] \cdot [\lim_{x\to\infty} g(x)/h(x)] \\ &= \lim_{x\to\infty} [f(x)/g(x)] \cdot [g(x)/h(x)] \\ &= \lim_{x\to\infty} f(x)/h(x), \end{aligned}$$

so $f \sim h$. ∎

**Problem 2.** Recall that for functions $f, g$ on the natural numbers, $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|\,. \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest natural number*, $c$, and for that smallest $c$, the *smallest corresponding natural number* $n_0$ ensuring that condition (1) applies.

**(a)** $f(n) = n^2, g(n) = 3n$.

$f = O(g)$ YES NO If YES, $c =$ ______, $n_0$ = ______

**Solution.** NO. ∎

$g = O(f)$ YES NO If YES, $c =$ ______, $n_0$ = ______

**Solution.** YES, with $c = 1$, $n_0 = 3$, which works because $3^2 = 9, 3 \cdot 3 = 9$. ■

**(b)** $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$ YES NO If YES, $c =$ _______, $n_0$ = _______

**Solution.** YES, with $c = 1, n_0 = 0$ (because $|f(n)| < 3$). ■

$g = O(f)$ YES NO If YES, $c =$ _______, $n_0$ = _______

**Solution.** YES, with $c = 2, n_0 = 15$.

Since $\lim_{n\to\infty} f(n) = 3$, the smallest possible $c$ is 2. For $c = 2$, the smallest possible $n_0 = 15$ which follows from the requirement that $2f(n_0) \geq 4$. ■

**(c)** $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$ YES NO If yes, $c =$ _______ $n_0$ = _______

**Solution.** NO, because $f(2n) = 1$, which rules out $g = O(f)$ since $g = \Theta(n)$. ■

$g = O(f)$ YES NO If yes, $c =$ _______ $n_0$ = _______

**Solution.** NO, because $f(2n + 1) = n^2 + 1 \neq O(n)$ which rules out $f = O(g)$. ■

**Problem 3.** Indicate which of the following holds for each pair of functions $(f(n), g(n))$ in the table below. Assume $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Be prepared to justify your answers.

| $f(n)$ | $g(n)$ | $f = O(g)$ | $f = o(g)$ | $g = O(f)$ | $g = o(f)$ | $f = \Theta(g)$ | $f \sim g$ |
|---|---|---|---|---|---|---|---|
| $2^n$ | $2^{n/2}$ | | | | | | |
| $\sqrt{n}$ | $n^{\sin n\pi/2}$ | | | | | | |
| $\log(n!)$ | $\log(n^n)$ | | | | | | |
| $n^k$ | $c^n$ | | | | | | |
| $\log^k n$ | $n^\epsilon$ | | | | | | |

**Solution.**

| $f(n)$ | $g(n)$ | $f = O(g)$ | $f = o(g)$ | $g = O(f)$ | $g = o(f)$ | $f = \Theta(g)$ | $f \sim g$ |
|---|---|---|---|---|---|---|---|
| $2^n$ | $2^{n/2}$ | no | no | yes | yes | no | no |
| $\sqrt{n}$ | $n^{\sin n\pi/2}$ | no | no | no | no | no | no |
| $\log(n!)$ | $\log(n^n)$ | yes | no | yes | no | yes | yes |
| $n^k$ | $c^n$ | yes | yes | no | no | no | no |
| $\log^k n$ | $n^\epsilon$ | yes | yes | no | no | no | no |

Following are some hints on deriving the table above:

(a) $\dfrac{2^n}{2^{n/2}} = 2^{n/2}$ grows without bound as $n$ grows—it is not bounded by a constant.

(b) When $n$ is even, then $n^{\sin n\pi/2} = 1$. So, no constant times $n^{\sin n\pi/2}$ will be an upper bound on $\sqrt{n}$ as $n$ ranges over even numbers. When $n \equiv 1 \mod 4$, then $n^{\sin n\pi/2} = n^1 = n$. So, no constant times $\sqrt{n}$ will be an upper bound on $n^{\sin n\pi/2}$ as $n$ ranges over numbers $\equiv 1 \mod 4$.

(c)

$$\begin{aligned}
\log(n!) &= \log \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \pm c_n && (2)\\
&= \log n + n(\log n - 1) \pm d_n && (3)\\
&\sim n \log n && (4)\\
&= \log n^n.
\end{aligned}$$

where $a \le c_n, d_n \le b$ for some constants $a, b \in \mathbb{R}$ and all $n$. Here equation (2) follows by taking logs of Stirling's formula, (3) follows from the fact that the log of a product is the sum of the logs, and (4) follows because any constant, $\log n$, and $n$ are all $o(n \log n)$ and hence so is their sum.

(d) *Polynomial* growth versus *exponential* growth.

(e) *Polylogarithmic* growth versus *polynomial* growth.

■

**Problem 4.** It is a standard fallacy to think that given $n$ quantities each of which is $O(1)$, their sum would have to be $O(n)$.

Namely, let $f_1, f_2, \ldots$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{N}$, and let

$$S(n) ::= \sum_{i=1}^{n} f_i(n).$$

Then given that $f_i = O(1)$ for every $f_i$ in the sequence, we can try to argue as follows:

$$S(n) = \sum_{i=1}^{n} f_i(n) = \sum_{i=1}^{n} O(1) = n \cdot O(1) = O(n).$$

This informal argument may seem plausible, but is fundamentally flawed because it treats O(1) as some kind numerical quantity. In fact, we ask you to show that there is no way to determine how fast the sum, $S(n)$, may grow.

Namely, let $g$ be any function on $\mathbb{N}$. Explain how to define a sequence of functions $f_1, f_2, \ldots$ such that each $f_i = O(1)$, but $S$ is not $O(g)$. *Hint:* Let $f_i(n) ::= 1 + ig(i)$.

**Solution.** Pick $f_i$ to be the constant function $i(1+g(i))$. That is,

$$f_i(n) ::= i(1+g(i)),$$

for all $n$. Since $f_i$ is a constant function, it is $O(1)$. But

$$S(n)\sum_{i=1}^{n} f_i(n) \geq f_n(n) = n(1+g(n)),$$

so $g = o(S)$ and therefore $S \neq O(g)$. ■

## Asymptotic Notations

For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say $f$ is *asymptotically equal* to $g$, in symbols,

$$f(x) \sim g(x)$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 1.$$

For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say $f$ is *asymptotically smaller* than $g$, in symbols,

$$f(x) = o(g(x)),$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 0.$$

Given functions $f, g : \mathbb{R} \mapsto \mathbb{R}$, with $g$ nonnegative, we say that[1]

$$f = O(g)$$

iff

$$\limsup_{x\to\infty} |f(x)| / g(x) < \infty.$$

An alternative, equivalent, definition is

$$f = O(g)$$

iff there exists a constant $c \geq 0$ and an $x_0$ such that for all $x \geq x_0$, $|f(x)| \leq cg(x)$.

Finally, we say

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \wedge g = O(f).$$

[1]

$$\limsup_{x\to\infty} h(x) ::= \lim_{x\to\infty} \text{lub}_{y\geq x} h(y).$$