

Solutions to In-Class Problems Week 8, Mon.

Problem 1. Given a simple graph G , we apply the following operation to the graph: pick two vertices $u \neq v$ such that either

1. there is an edge of G between u and v and there is also a path from u to v which does *not* include this edge; in this case, delete the edge $\{u, v\}$.
2. or, there is no path from u to v ; in which case, add the edge $\{u, v\}$.

We keep repeating these operations until it is no longer possible to find two vertices $u \neq v$ to which an operation applies.

Assume the vertices of G are the integers $1, 2, \dots, n$ for some $n \geq 2$. This procedure can be modelled as a state machine whose states are all possible simple graphs with vertices $1, 2, \dots, n$. The start state is G , and the final states are the graphs on which no operation is possible.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\{1, 2\}, \{3, 4\}\}$$

What are the possible final states reachable from start state G ? Draw them.

Solution. It's not possible to delete any edge. The procedure can only add an edge connecting exactly one of vertices 1 or 2 to exactly one of vertices 3 or 4, and then terminate. So there are four possible final states ■

(b) For any state, G' , let e be the number of edges in G' , c be the number of connected components it has, and s be the number of simple cycles. For each of the derived variables below, indicate the *strongest* of the properties that it is guaranteed to satisfy, no matter what the starting graph G is. The choices for properties are: *constant*, *strictly increasing*, *strictly decreasing*, *weakly increasing*, *weakly decreasing*, *none of these*. The derived variables are

(i) $e - s$

Solution. weakly increasing ■

(ii) $c + e$ **Solution.** weakly decreasing ■(iii) $3c + 2e$ **Solution.** strictly decreasing ■(iv) $c + s$ **Solution.** strictly decreasing ■(v) (c, e) , partially ordered coordinatewise (the *product* partial order).**Solution.** none of these ■

(c) Choose a derived variable from above and prove that it is strictly decreasing in some well-founded partial order. Conclude that the procedure terminates.

Solution. To show that the variable (ii) strictly decreases, note that the rule for deleting an edge ensures that the connectedness relation does not change, so neither does the number of connected components c . Meanwhile the number of edges e decreases by one when an edge is deleted. Therefore the variable $3c + 2e$ decreases by 2. The rule for adding an edge ensures that the number of connected components c decreases by one and the number of edges e increases by one. Therefore the variable $3c + 2e$ decreases by 1.

To show that the variable (iii) strictly decreases, note that the rule for deleting an edge ensures that the number of connected components c does not change and the number of simple cycles s decreases by n , where $n \geq 1$. Therefore the variable $c + s$ decreases by n . The rule for adding an edge ensures that the number of connected components c decreases by one and the number of simple cycles s does not change. Therefore the variable $c + s$ decreases by one.

To show that the lexicographically ordered (c, e) strictly decreases, note that the rule for deleting an edge ensures that the number of connected components c does not change and the number of edges e decreases by one. The rule for adding an edge ensures that the number of connected components c decreases by one. ■

(d) Prove that any final state must be a tree on the vertices.

Solution. We use the characterization of a tree as a cycle-free, connected, simple graph.

A final state must be connected, because otherwise there would be two vertices with no path between them, and then a transition adding the edge between them would be possible, contradicting finality of the state.

A final state can't have a cycle, because deleting any edge on the cycle would be a possible transition. ■

Problem. See if you can come up with a stable marriage assignment given the following preferences. You are not expected to know/remember the Dating Protocol that solves this problem and which is about to be covered in lecture. (And if you do remember the protocol, don't spoil your teammates' fun by telling them.)

<i>boys</i>	<i>girls</i>
1 : <i>CBEAD</i>	<i>A</i> : 35214
2 : <i>ABECD</i>	<i>B</i> : 52143
3 : <i>DCBAE</i>	<i>C</i> : 43512
4 : <i>ACDBE</i>	<i>D</i> : 12345
5 : <i>ABDEC</i>	<i>E</i> : 23415

Solution. See Slides 8M ([pdf](#)). ■

Problem 2. Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
David:	HP, Bellcore, AT&T, Draper
Jelani:	AT&T, Bellcore, Draper, HP
Hanson:	HP, Draper, AT&T, Bellcore
Sayan:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Sayan, David, Hanson, Jelani
Bellcore:	Hanson, Jelani, David, Sayan
HP:	Sayan, Hanson, David, Jelani
Draper:	Jelani, Sayan, Hanson, David

(a) Use the Mating Algorithm to find *two* stable assignments of Students to Companies.

Solution. Treat Students as Boys and the result is the following assignment:

Student	Companies	Rank in the original list
David:	Bellcore	2
Jelani:	AT&T	1
Hanson:	HP	1
Sayan:	Draper	1

Treat Companies as Boys and the result is the following assignment:

Company	Students	Rank in the original list
AT&T:	David	2
Bellcore:	Jelani	2
HP:	Hanson	2
Draper:	Sayan	2

■

Problem 3. Prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Course Notes.)

Solution. *Proof.* We have that the following predicate, P , is an invariant of the Mating ritual:

For every girl, G , and every boy, B , if G is crossed off B 's list, then G has a favorite suitor and she prefers him over B .

Now let Bob be some boy and Carole be any girl that he is *not* married to on the last day of the Mating ritual. We will prove that Bob and Carole are not a rogue couple. Since Bob was an arbitrary boy, it follows that no boy is part of a rogue couple. Hence the marriages on the last day are stable.

To prove the claim, we consider two cases:

Case 1. Carole is not on Bob's list. Then since invariant P holds, we know that Carole prefers her husband to Bob. So she's not going to run off with Bob: the claim holds in this case.

Case 2. Otherwise, Carole is on Bob's list. But since Bob is not married to Carole, he must have chosen to serenade his wife instead of Carole, so he must prefer his wife. So he's not going to run off with Carole: the claim also holds in this case. □

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