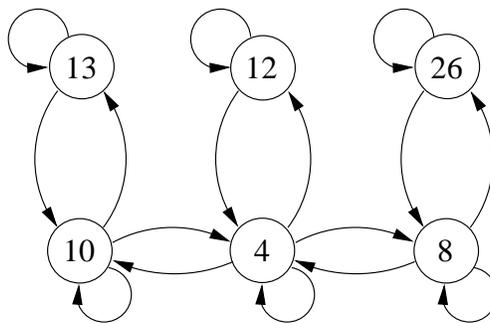


Solutions to In-Class Problems Week 4, Wed.

Problem 1. Recently MIT students have been taking a hard look at the haphazard building layout, and have been asking some hard questions. As always they know they can use their superior mathematical skills to get some real answers to those hard questions.

They decide to express the MIT building layout as a relation. Let C be the set of all building numbers and let R be the relation on the set C such that $(a, b) \in R$ if building a and building b are physically adjacent and there is a door between a and b (more importantly, one doesn't have to go outside to get from a to b). Note that if $(a, b) \in R$, then (b, a) is also in R , so R is a symmetric relation. For convenience, they also define a building to be related to itself, so $(a, a) \in R$.

(a) For this part only, let C be the set of MIT buildings 10,13,12,4,8,26. Then R looks like this:



- R^2 consists of all pairs of buildings that are connected by traversing exactly 2 edges. Compute R^2 .
- $R^{\leq 2}$ consists of all pairs of buildings that are connected by traversing at most 2 edges. Compute $R^{\leq 2}$.
- R^3 consists of all pairs of buildings that are connected by traversing exactly 3 edges. Compute R^3 .
- If all self-loops were removed (i.e., all edges of the form (a, a)), would $R^2 = R^{\leq 2}$?

Solution. Since there are self-loops in the connectivity graph (each building is connected to itself), R^2 includes everything which is in R as well. For example, $(13, 10)$ which is in R must also be in

R^2 because you can go from building 13 to building 10 “via 2 edges” by just going around building 13 once and then moving on to building 10. That said, we conclude that

$$R^2 = R \cup \{(13, 4), (4, 13), (10, 12), (12, 10), (10, 8), (8, 10), (12, 8), (8, 12), (4, 26), (26, 4)\}$$

$$R^{\leq 2} = R^2.$$

Similarly, R^3 contains everything which is in R^2 (because of the self-loops), plus all the extra pairs that we get by allowing connections via *three* edges. So we have:

$$R^3 = R^2 \cup \{(13, 12), (12, 13), (10, 26), (26, 10), (13, 8), (8, 13), (26, 12), (12, 26)\}$$

If the self-loops were removed, R^2 would not contain $(13, 13)$ but $R^{\leq 2}$ would. ■

(b) We would like to connect the buildings so that for any pair of buildings (a, b) , either one can reach b from a or one can reach a from b . The MIT administration wants to keep the number of connections between buildings as small as possible. In other words, MIT wants the size, $|R|$, of R to be as small as possible. What is the smallest R that will satisfy the MIT students? Is the smallest R unique?

Solution. The smallest graph such that all the buildings are connected would be to connect them in a straight line. The number of edges in that graph is $n - 1$. How do we know that this is the smallest number of edges? We will learn a theorem that says that the smallest connected graph with vertices n has at least $n - 1$ edges. Each edge contributes two pairs therefore $|R| = 2(n - 1)$. The smallest graph is not unique, for starters we can connect the buildings in a different order in the line. But there are many other connected graphs with n vertices and $n - 1$ edges (any tree with n nodes in fact satisfies this constraint, as we shall soon see). ■

Problem 2. (a) What are the maximal and minimal elements, if any, of the set, \mathbb{N} , of all natural numbers under divisibility? Is there a minimum or maximum element?

Solution. The minimum (and therefore unique minimal) element is 1 since 1 divides all natural numbers. The maximum (and therefore unique maximal) element is 0 since all numbers divide 0. ■

(b) What are the minimal and maximal elements, if any, of the set of integers ≥ 2 under divisibility?

Solution. All prime numbers are minimal elements, since no numbers divide them. Since there is more than one minimal element, there can't be a minimum element.

There is no maximal element, because for any $n \geq 2$, there is a “larger” number under the divisibility partial order, namely, mn , for any $m > 1$. ■

Problem 3. (a) Describe a sequence consisting of the integers from 1 to 10,000 in some order so that there is no increasing or decreasing subsequence of size 101.

Solution.

$$[100 \rightarrow 1][200 \rightarrow 101][300 \rightarrow 201] \dots [10,000 \rightarrow 9901]$$

■

(b) What is the size of the longest chain that is guaranteed to exist in any partially ordered set of n elements? What about the largest antichain?

Solution. Chain size is 1 in the “discrete” partial order in which every two distinct elements are incomparable. Antichain size is 1 if the partial order is total. ■

(c) Describe a partially ordered set that has no minimal or maximal elements.

Solution. \mathbb{Z}, \mathbb{R} , etc. ■

(d) Describe a partially ordered set that has a *unique minimal* element, but no minimum element.

Solution. $\mathbb{Z} \cup i$ where i is a root of -1 , under the usual order \mathbb{Z} . So i is incomparable to everything but itself, and is therefore minimal. ■

Problem 4. A pair of 6.042 TAs, Zardosht and Ching, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Ching’s copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Zardosht’s cat, Emilios - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.

7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.
8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

(a) Express the information in the task list using some type of graph (label the vertices to reflect task lengths).

Solution. The information in the table is represented in the Figure ?? in the form of a directed acyclic graph. Each vertex represents a task, and the weight of a vertex is the completion time. Each directed edge represents a dependency between tasks.

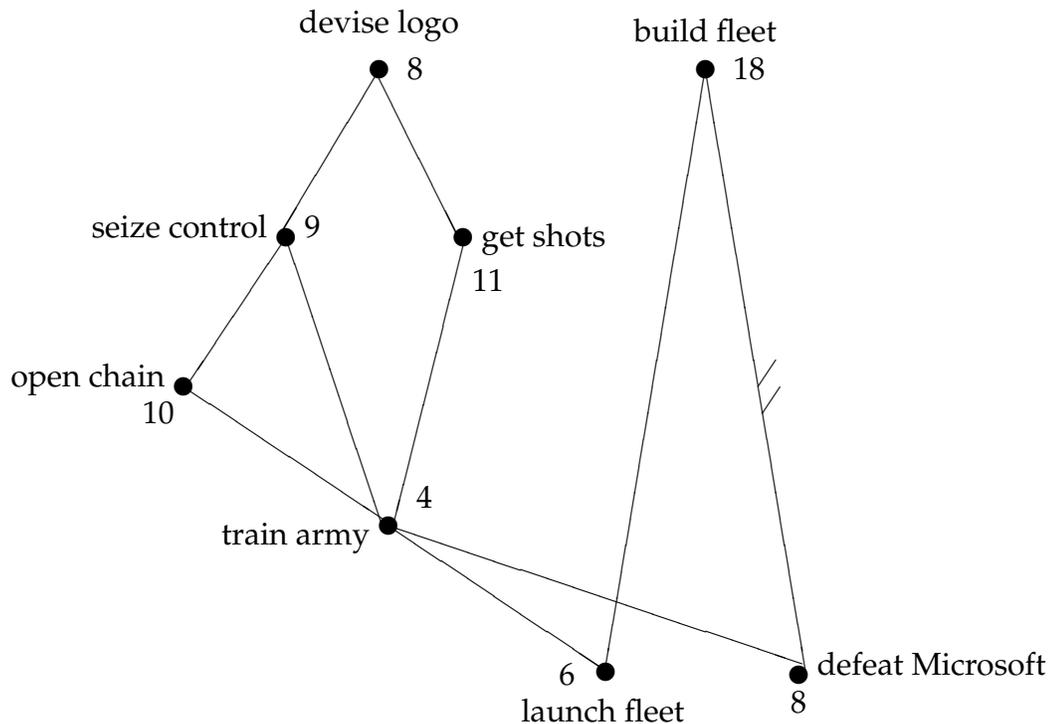


Figure 1: Graph representing the task precedence constraints.

(b) Give some valid order in which the tasks might be completed.

Solution. We can easily find several of them. The most natural one is valid, too: #1, #2, #3, #4, #5, #6, #7, #8. ■

Zardosht and Ching want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.

- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Zardosht cannot work on building a fleet for a few days, run get shots for Emilios, and then return to building the fleet.

(c) Zardosht and Ching want to know how long conquering the galaxy will take. Ching suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

Solution.

$$\frac{8 + 18 + 9 + 11 + 10 + 4 + 6 + 8}{2} = 37 \text{ days}$$

If working together and interrupting work on a task were permitted, then this answer would be correct. However, the rules may prevent Zardosht and Ching from both working all the time. ■

(d) Zardosht proposes a different method for determining the duration of their project. He suggests looking at the duration of the “critical path”, the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

Solution. The longest sequence of tasks is devising a logo (8 days), seizing the U. N. (9 days), opening a Starbucks (10 days), training the army (4 days), and then defeating Microsoft (8 days). Since these tasks must be done sequentially, galactic conquest will require at least 39 days.

If there were enough workers, this answer would be correct; however, with only two workers, Zardosht and Ching may be unable to make progress on the critical path every day. ■

(e) What is the minimum number of days that Zardosht and Ching need to conquer the galaxy? No proof is required.

Solution. 40 days. Tasks could be divided as follows:

Ching: #1 (days 1-8), #3 (days 9-17), #4 (days 18-28), #8 (days 33-40).

Zardosht: #2 (days 1-18), #5 (days 19-28), #6 (days 29-32), #7 (days 33-38). ■

Appendix

Definition. Let \preceq be a weak (reflexive) partial order on a set, A .

- An element $a \in A$ is a *lower bound* for a subset, $S \subseteq A$ iff $a \preceq s$ for every $s \in S$. Similarly, an element $a \in A$ is an *upper bound* for a subset, $S \subseteq A$ iff $s \preceq a$ for every $s \in S$.
- An element $a \in A$ is the *minimum* element iff a is a lower bound on A . Similarly, an element $a \in A$ is the *maximum* element iff a is an upper bound on A .

- An element $a \in A$ is the *greatest lower bound (glb)* of a subset, $S \subseteq A$ iff a is a lower bound for S , and if $b \in A$ is also a lower bound for S , then $b \preceq a$. Similarly for *least upper bound (lub)*.
- An element $a \in A$ is *minimal* iff there is no element in A that is $\preceq a$ except a itself. Similarly, an element $a \in A$ is *maximal* iff there is no element in A that is $\succeq a$ except a itself.
- Elements $a, b \in A$ are *comparable* iff either $a \preceq b$ or $b \preceq a$. Two elements are *incomparable* iff they are not comparable.
- A subset, $S \subseteq A$ is *totally ordered* iff every two distinct elements in S are comparable.
- A *chain* is a totally ordered subset of A .
- An *antichain* is a subset of A , such that no two elements in it are comparable.