Solutions to In-Class Problems Week 4, Mon.

Problem 1. In each case, say whether or not R is a equivalence relation on A. If it is an equivalence relation, what are the equivalence classes and how many equivalence classes are there?

(a) $R ::= \{(x, y) \in W \times W \mid$ the words x and y start with the same letter$\}$ where W is the set of all words in the 2001 edition of the Oxford English dictionary.

Solution. R is an equivalence relation since it is reflexive, symmetric, and transitive. The equivalence class of x with respect to R is the set $[x]_R =$ the set of words y, such that y has the same first letter as x. There are 26 equivalence classes, one for each letter of the English alphabet. ■

(b) $R ::= \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common$\}$.

Solution. S is reflexive and symmetric, but it is not transitive. Therefore, S is not an equivalence relation. For example, let w_1 be the word "scream," let w_2 be the word "and," and let w_3 be the word "shout." Then $w_3 S w_1$, and $w_1 S w_2$, but it is **not** the case that $w_3 S w_2$. ■

(c) $R = \{(x, y) \in W \times W$ and the word x comes before the word y alphabetically$\}$.

Solution. R is not reflexive but it is transitive and antisymmetric. It is not an equivalence relation, but it is a partial order. ■

(d) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R}$ and $|x| \leq |y|\}$.

Solution. R is reflexive and transitive. It is not symmetric. It is not antisymmetric either. As a counterexample, $|-3| \leq |3|, |3| \leq |-3|$, but $3 \neq -3$. ■

(e) $R = \{(x, y) \in B \times B$, where B is the set of all bit strings and x and y have the same number of 1s.$\}$

Solution. R is reflexive, symmetric and transitive, and therefore an equivalence relation. There is an equivalence class for each natural number corresponding to bit strings with that number of 1s. ■

Problem 2.

False Claim. *Suppose R is a relation on A. If R is symmetric and transitive, then R is reflexive.*

(a) Give a counter-example to the claim.

Solution. The simplest counterexample is to let R be the empty relation on some nonempty set A. This R is vacuously symmetric and transitive, but obviously not reflexive.

A slightly less trivial example is

$$R ::= \{(a, a), (a, b), (b, a), (b, b)\}$$

on the set $A ::= \{a, b, c\}$. It is not reflexive because (c, c) is not in R. ■

(b) Find the flaw in the following proof of the claim:

False proof. Let x be an arbitrary element of A. Let y be any element of A such that xRy. Since R is symmetric, it follows that yRx. Then since xRy and yRx, we conclude by transitivity that xRx. Since x was arbitrary, we have shown that $\forall x \in A\ (xRx)$, so R is reflexive. □

Solution. The flaw is assuming that y exists. It is possible that there is an $x \in A$ that is not related by R to anything. No such R will be reflexive.

Note that the theorem can be fixed: R restricted to its domain of definition is reflexive, and hence an equivalence relation. ■

Problem 3. Verify that each of the following relations is a partial order by describing a function, g, such that the relation is defined by g according to the Definition 4.2 in the Appendix. For each, is it a total order?

(a) The relation, $<$, on $\mathbb{R}$.

Solution. Define $g(r) ::= \{t \in \mathbb{R} \mid t < r\}$. It follows that

$$r_1 < r_2 \quad \text{iff} \quad g(r_1) \subset g(r_2)$$

so $<$ satisfies the condition (3) on R that defines partial orders. Likewise, the relation, $\leq$, is a partial order because

$$r_1 \leq r_2 \quad \text{iff} \quad r_1 < r_2,$$

for all reals $r_1 \neq r_2$. ■

(b) The superset relation, $\supseteq$, on $\mathcal{P}(B)$ for a set, B.

Solution. Define $g(a) ::= \overline{a} ::= B - \{a\}$, and note that for $a_1 \neq a_2 \in \mathcal{P}(B)$,

$$\begin{aligned} a_1 \supseteq a_2 \quad &\text{iff} \quad a_1 \supset a_2 && \text{(since } a_1 \neq a_2\text{)} \\ &\text{iff} \quad \overline{a_1} \subset \overline{a_2} && \text{(basic set theory)} \\ &\text{iff} \quad g(a_1) \subset g(a_2) && \text{(def of } g\text{).} \end{aligned}$$

■

(c) The "divides" relation on natural numbers.

Solution. Let $g(a) ::=$ the set of natural number that divide a. ■

Problem 4. Suppose you are given the description of an equivalence relation and want to cut down on the number of pairs that are stored without losing any information. For example, here are the pairs of an equivalence relation on a set of integers:

$$G ::= \{11, 33, 44, 55, 66, 77, 13, 31, 45, 54, 47, 74, 57, 75\}.$$

where for readability, we've written "mk" to designate the pair (m, k).

To start, we know that if we have mk, then we necessarily also have km, so there's no need to keep both. This lets us cut down to:

$$\{11, 33, 44, 55, 66, 77, 13, 45, 47, 57\}.$$

Also, if we have 45 and 57, we don't need 47, since that will necessarily be there (by transitivity), so we can further cut down to

$$\{11, 33, 44, 55, 66, 77, 13, 45, 57\}$$

In addition, as long as we keep some pair in which k appears, we don't need the pair kk. This lets us cut down to the pairs

$$\{66, 13, 45, 57\}. \tag{1}$$

These pairs are all that are needed to determine the entire original equivalence relation, G. Moreover, the set (1) of these pairs is *minimal* with this property; this means that if any pair was removed from the set, it wouldn't determine the relation any more.

(a) Describe another couple of minimal sets of pairs that determine the relation.

Solution.

$$\{66, 31, 45, 57\}, \{66, 13, 47, 57\}, \text{ and lots more} \ldots.$$

■

(b) Here are the pairs that are left after some unnecessary pairs have been removed from the description of an equivalence relation, E. What is the domain of E? What are the equivalence classes of E?

$$15, 20, 40, 57, 68, 79, 9a, bb, c3$$

Solution.

$$\text{domain of } E = \{0123456789abc\}$$
$$\text{equivalence classes of } E = \{024\}\,\{1579a\}\,\{3c\}\,\{68\}\,\{b\}.$$

■

(c) On a domain of n elements, what is the smallest number of pairs that could determine an equivalence relation?

Solution. $\lceil n/2 \rceil$ by having all blocks of size 2, except for one block of size 1 if n is odd. ■

(d) Suppose you have an equivalence relation on a domain of size n with k equivalence classes, with no classes of just one element. Then every minimal set of pairs has the same size. What is that size? Explain.

Solution. It takes $n - k$ pairs. We'll let you convince yourself that $c - 1$ is the minimum number of pairs to "connect up" all the elements and thereby determine an equivalence class of size $c > 1$. (We'll prove this carefully in couple of weeks when we study tree structures.) So if there are k classes of sizes $c_1, \ldots, c_k$, you need

$$(c_1 - 1) + (c_2 - 1) + \cdots + (c_k - 1) = (c_1 + c_2 + \cdots + c_k) - k = n - k$$

pairs. ■

Appendix

Equivalence

Definition 4.1. A binary relation, R, on a set, A, is an *equivalence relation* iff there is a function, f, with domain A, such that

$$a_1 \, R \, a_2 \quad \text{iff} \quad f(a_1) = f(a_2) \tag{2}$$

for all $a_1, a_2 \in A$.

Theorem. *A relation is an equivalence iff it is reflexive, symmetric and transitive.*

Partial Order

Definition 4.2. A relation, R, on a set, A, is a *partial order* providing there is a function, g, from A to some collection of sets such that

$$a_1 \, R \, a_2 \quad \text{iff} \quad g(a_1) \subset g(a_2), \tag{3}$$

for all $a_1 \neq a_2 \in A$.

Theorem. *A relation is a partial order iff it is transitive and antisymmetric.*

Relational Properties

A binary relation, R, on a set, A, is

- *reflexive* if aRa for every $a \in A$,
- *irreflexive* if aRa holds for no $a \in A$,
- *symmetric* if for every $a, b \in A$, aRb implies bRa,
- *antisymmetric* if for every $a \neq b \in A$, aRb implies $\neg(bRa)$,
- *asymmetric* if for every $a, b \in A$, aRb implies $\neg(bRa)$,
- *transitive* if for every $a, b, c \in A$, aRb and bRc implies aRc.