

Solutions to In-Class Problems Week 3, Wed.

Problem 1. Use induction to prove that

$$1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2. \quad (1)$$

for all $n \geq 1$.

Remember to formally

1. Declare proof by induction.
2. Identify the induction hypothesis $P(n)$.
3. Establish the base case.
4. Prove that $P(n) \Rightarrow P(n+1)$.
5. Conclude that $P(n)$ holds for all $n \geq 1$.

as in the five part template.

Solution. We proceed by induction. The induction hypothesis, $P(n)$, will be the equation (1).

Base case: First, we must show that $P(1)$ is true. This is immediate, since:

$$1^3 = \left(\frac{1(1+1)}{2} \right)^2$$

Inductive step: Next, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Assuming that $P(n)$ is true, we can reason as follows:

$$\begin{aligned} 1^3 + 2^3 + \dots + n^3 + (n+1)^3 &= \left(\frac{n(n+1)}{2} \right)^2 + (n+1)^3 \\ &= \left(\frac{(n+1)(n+2)}{2} \right)^2 \end{aligned}$$

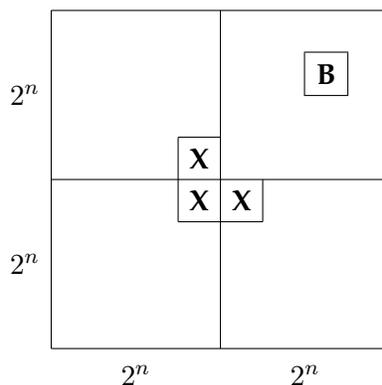
The first step uses the assumption $P(n)$, and the second uses only algebra. This shows that $P(n+1)$ is true. Therefore, $P(n)$ is true for all $n \geq 1$ by induction. ■

Problem 2. (a) Prove by induction that a $2^n \times 2^n$ courtyard with a 1×1 statue of Bill in *any position* can be covered with L-shaped tiles.

Solution. Let $P(n)$ be the proposition that for every location of Bill in a $2^n \times 2^n$ courtyard, there exists a tiling of the remainder.

Base case: $P(0)$ is true because Bill fills the whole courtyard.

Inductive step: Assume that $P(n)$ is true for some $n \geq 0$; that is, for every location of Bill in a $2^n \times 2^n$ courtyard, there exists a tiling of the remainder. Divide the $2^{n+1} \times 2^{n+1}$ courtyard into four quadrants, each $2^n \times 2^n$. One quadrant contains Bill (**B** in the diagram below). Place a temporary Bill (**X** in the diagram) in each of the three central squares lying outside this quadrant:



Now we can tile each of the four quadrants by the induction assumption. Replacing the three temporary Bills with a single L-shaped tile completes the job. This proves that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. The theorem follows as a special case.

This proof has two nice properties. First, not only does the argument guarantee that a tiling exists, but also it gives a recursive procedure for finding such a tiling. Second, we have a stronger result: if Bill wanted a statue on the edge of the courtyard, away from the pigeons, we could accommodate him! ■

(b) (Discussion Question) In part (a) we saw that it can be easier to prove a stronger theorem. Does this surprise you? How would you explain this phenomenon?

Solution. It might seem that it ought to be harder to prove a more general theorem than a less general one, but sometimes not. For example, the more general result might actually be easier because it involves fewer assumptions, and this can help in avoiding the complications of unnecessary hypotheses.

But for an induction proof in particular, using a more general induction hypothesis means we can make a stronger *assumption* in the induction step (namely, we can assume a stronger $P(n)$), which can make it easier to prove the conclusion of the induction step (namely, $P(n+1)$). ■