Massachusetts Institute of Technology
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Solutions to In-Class Problems Week 3, Fri.

Problem 1. Given an unlimited supply of 3 cent and 5 cent stamps, what postages are possible? Prove it using Strong Induction. *Hint:* Try some examples! Which postage values between 1 and 25 cents can you construct from 3 cent and 5 cent stamps?

Solution. Let's use our examples to first try to guess the answer and then try to prove it. Let's begin filling in a table that shows the values of all possible combinations of 3 and 5 cent stamps. The column heading is the number of 5 cent stamps and the row heading is the number of 3 cent stamps.

	0	1	2	3	4	5	...
0	0	5	10	15	20	25	...
1	3	8	13	18	23	...	
2	6	11	16	21	...		
3	9	14	19	24	...		
4	12	17	22	...			
5	15	20	...				
...	...	...					

Looking at the table, a reasonable guess is that the possible postages are 0, 3, 5, and 6 cents and every value of 8 or more cents. Let's try to prove this last part using strong induction.

Claim 1.1. *For all $n \geq 8$, it is possible to produce n cents of postage from 3¢ and 5¢ stamps.*

Now let's preview the proof. The induction hypothesis will be

$$P(n) \quad ::= \quad \text{if } n \geq 8, \text{ then } n¢ \text{ postage can be produced using 3¢ and 5¢ stamps} \qquad (1)$$

A proof by strong induction will have the same five-part structure as an ordinary induction proof. The base case, $P(0)$, won't be interesting because $P(n)$ is *vacuously* true for all $n < 8$.

In the inductive step we have to show how to produce $n+1$ cents of postage, assuming the strong induction hypothesis that we know how to produce k¢ of postage for all values of k between 8 and n. A simple way to do this is to let $k = n - 2$ and produce k¢ of postage; then add a 3¢ stamp to get $n + 1$ cents.

But we have to be careful; there is a pitfall in this method. If $n + 1$ is 8, 9 or 10, then we can not use the trick of creating $n + 1$ cents of postage from $n - 2$ cents and a 3 cent stamp. In these cases, $n - 2$ is less than 8. None of the strong induction assumptions help us make less than 8¢ postage. Fortunately, making $n + 1$ cents of postage in these three cases can be easily done directly.

Proof. The proof is by strong induction. The induction hypothesis, $P(n)$, is given by (1).

Base case: $n = 0$: $P(0)$ is true vacuously.

Inductive step: In the inductive step, we assume that it is possible to produce postage worth $8, 9, \ldots, n$ cents in order to prove that it is possible to produce postage worth $n + 1$ cents.

There are four cases:

1. $n + 1 < 8$: So $P(n + 1)$ holds vacuously.
2. $n + 1 = 8$: $P(n + 1)$ holds because we produce 8¢ postage using one 3¢ and one 5¢ stamp.
3. $n + 1 = 9$: $P(n + 1)$ holds by using three 3¢ stamps.
4. $n + 1 = 10$: $P(n + 1)$ holds by using two 5¢ stamps.
5. $n + 1 > 10$: We have $n \geq 10$, so $n - 2 \geq 8$ and by strong induction we may assume we can produce exactly $n - 2$ cents of postage. With an additional 3¢ stamp we can therefore produce $n + 1$ cents of postage.

So in every case, $P(0) \wedge P(1) \wedge \ldots P(n) \longrightarrow P(n + 1)$. By strong induction, we have concluded that $P(n)$ is true for all $n \in \mathbb{N}$. □

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Problem 2. Use the Well-ordering Principle to prove that there is no solution over the positive integers to the equation:

$$4a^3 + 2b^3 = c^3.$$

Solution. We use contradiction and the well-ordering principle. Let S be the set of all positive integers, a, such that there exist positive integers, b, and, c, that satisfy the equation.

Assume for the purpose of obtaining a contradiction that S is nonempty. Then S contains a smallest element, a_0, by the well-ordering principle. By the definition of S, there exist corresponding positive integers, b_0, and, c_0, such that:

$$4a_0^3 + 2b_0^3 = c_0^3$$

The left side of this equation is even, so c_0^3 is even, and therefore c_0 is also even. Thus, there exists an integer, c_1, such that $c_0 = 2c_1$. Substituting into the preceding equation and then dividing both sides by 2 gives:

$$2a_0^3 + b_0^3 = 4c_1^3$$

Now b_0^3 must be even, so b_0 is even. Thus, there exists an integer, b_1, such that $b_0 = 2b_1$. Substituting into the preceding equation and dividing both sides by 2 again gives:

$$a_0^3 + 4b_1^3 = 2c_1^3$$

From this equation, we know that a_0^3 is even, so a_0 is also even. Thus, there exists an integer, a_1, such that $a_0 = 2a_1$. Substituting into the previous equation one last time and dividing by 2 one last time gives:

$$4a_1^3 + 2b_1^3 = c_1^3$$

Evidently, $a = a_1$, $b = b_1$, and $c = c_1$ is another solution to the original equation, and so a_1 is an element of S. But this is a contradiction, because $a_1 < a_0$ and a_0 was defined to be the smallest element of S. Therefore, our assumption was wrong, and the original equation has no solutions over the positive integers.

This argument is quite similar to the proof that $\sqrt{2}$ is irrational. In fact, looking back, we implicitly relied on the Well-ordering Principle in that proof when we claimed that a rational number could be written as a fraction in *lowest terms*. We've been using the Well-ordering Principle on the sly from early on! ■