Solutions to In-Class Problems Week 2, Wed.

Problem 1. For each of the logical formulas, indicate whether or not it is true when the domain of discourse is $\mathbb{N}$ (the natural numbers 0, 1, 2, ...), $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers).

$$\begin{array}{rlrcl} & \exists x & (x^2 & = & 2) \\ \forall x & \exists y & (x^2 & = & y) \\ \forall y & \exists x & (x^2 & = & y) \\ \forall x \neq 0 & \exists y & (xy & = & 1) \\ \exists x & \exists y & (x+2y & = & 2) \wedge (2x+4y=5) \end{array}$$

Solution.

Statement	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
$\exists x\ (x^2 = 2)$	f	f	f	$t\ (x = \sqrt{2})$	t
$\forall x \exists y\ (x^2 = y)$	t	t	t	$t\ (y = x^2)$	t
$\forall y \exists x\ (x^2 = y)$	f	f	f	f (take $y < 0$)	t
$\forall x \neq 0 \exists y\ (xy = 1)$	f	f	t	$t\ (y = 1/x)$	t
$\exists x \exists y\ (x + 2y = 2) \wedge (2x + 4y = 5)$	f	f	f	f	f

■

Problem 2. The goal of this problem is to translate some assertions about binary strings into logic notation. The domain of discourse is the set of all finite-length binary strings: λ, 0, 1, 00, 01, 10, 11, 000, 001, ... (Here λ denotes the empty string.) In your translations, you may use all the ordinary logic symbols (including =), variables, and the binary symbols `0`, `1` denoting 0, 1.

A string like 01x0y of binary symbols and variables denotes the *concatenation* of the symbols and the binary strings represented by the variables. For example, if the value of x is `011` and the value of y is `1111`, then the value of 01x0y is the binary string `0101101111`.

Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as we do in the definition of the predicate NO-1S below).

Meaning	Formula	Name
x is a prefix of y	$\exists z\ (xz = y)$	$\text{PREFIX}(x, y)$
x is a substring of y	$\exists u \exists v\ (uxv = y)$	$\text{SUBSTRING}(x, y)$
x is empty or a string of 0's	$\neg\text{SUBSTRING}(1, x)$	$\text{NO-1S}(x)$

(a) x consists of three copies of some string.

Solution. $\exists y\ (x = yyy)$ ■

(b) x is an even-length string of 0's.

Solution. $\text{NO-1S}(x) \wedge \exists y\ (x = yy)$ ■

(c) x does not contain both a 0 and a 1.

Solution. $\neg[\text{SUBSTRING}(0, x) \wedge \text{SUBSTRING}(1, x)]$

■

(d) x is the binary representation of $2^k + 1$ for some integer $k \geq 0$.

Solution. $(x = 10) \vee (\exists y\ (x = 1y1 \wedge \text{NO-1S}(y)))$ ■

(e) An elegant, slightly trickier way to define $\text{NO-1S}(x)$ is:

$$\text{PREFIX}(x, 0x). \quad (*)$$

Explain why (*) is true only when x is a string of 0's.

Solution. Prefixing x with 0 rightshifts all the bits. So the nth symbol of x shifts into the $(n + 1)$st symbol of $0x$. Now for x to be a prefix of $0x$, the $n + 1$st symbol of $0x$ must match the $(n + 1)$st symbol of x. So if x satisfies (*), the nth and $(n + 1)$st symbols of x must match. This holds for all $n > 0$ up to the length of x, that is, *all* the symbols of x must be the same. In addition, if $x \neq \lambda$, it must start with 0. Therefore, if x satisfies (*), all its symbols must be 0's.

Note that it's easy to see, conversely, that if $x = \lambda$ or x is all 0's, then of course it satisfies (*). ■

Problem 3. A media tycoon has an idea for an all-news television network called LNN: The Logic News Network. Each segment will begin with a definition of the domain of discourse and a few predicates. The day's happenings can then be communicated concisely in logic notation. For example, a broadcast might begin as follows:

"THIS IS LNN. The domain of discourse is $\{\text{Bill}, \text{Monica}, \text{Ken}, \text{Linda}, \text{Betty}\}$. Let $D(x)$ be a predicate that is true if x is deceitful. Let $L(x, y)$ be a predicate that is true if x likes y. Let $G(x, y)$ be a predicate that is true if x gave gifts to y."

Complete the broadcast by translating the following statements into logic notation.

(a) If neither Monica nor Linda is deceitful, then Bill and Monica like each other.

Solution.

$$(\neg(D(\text{Monica}) \vee D(\text{Linda}))) \longrightarrow (L(\text{Bill}, \text{Monica}) \wedge L(\text{Monica}, \text{Bill}))$$

■

(b) Everyone except for Ken likes Betty, and no one except Linda likes Ken.

Solution.

$$\forall x \; (x = \text{Ken} \wedge \neg L(x, \text{Betty})) \vee (x \neq \text{Ken} \wedge L(x, \text{Betty})) \wedge$$
$$\forall x \; (x = \text{Linda} \wedge L(x, \text{Ken})) \vee (x \neq \text{Linda} \wedge \neg L(x, \text{Ken}))$$

■

(c) If Ken is not deceitful, then Bill gave gifts to Monica, and Monica gave gifts to someone.

Solution.

$$\neg D(\text{Ken}) \longrightarrow (G(\text{Bill}, \text{Monica}) \wedge \exists \, x G(\text{Monica}, x))$$

■

(d) Everyone likes someone and dislikes someone else.

Solution.

$$\forall x \exists y \exists z \; (y \neq z) \wedge L(x, y) \wedge \neg L(x, z)$$

■

(e) How could you express "Everyone except for Ken likes Betty" using just propositional connectives *without* using any quantifiers ($\forall, \exists$)? Can you generalize to explain how *any* logical formula over this domain of discourse can be expressed without quantifiers? How big would the formula in the previous part be if it was expressed this way?

Solution.

$$L(\text{Bill}, \text{Betty}) \wedge L(\text{Monica}, \text{Betty}) \wedge L(\text{Linda}, \text{Betty}) \wedge L(\text{Betty}, \text{Betty}) \wedge \neg L(\text{Ken}, \text{Betty})$$

In general, quantifiers can be eliminated by treating $\forall x\ P(x)$ as an abbreviation for

$$P(\text{Bill}) \wedge P(\text{Monica}) \wedge P(\text{Ken}) \wedge P(\text{Linda}) \wedge P(\text{Betty}),$$

and $\exists x\ P(x)$ as an abbreviation for

$$P(\text{Bill}) \vee P(\text{Monica}) \vee P(\text{Ken}) \vee P(\text{Linda}) \vee P(\text{Betty}).$$

Expanded this way, the three-quantifier formula of the previous part would expand by a factor of $5 \times 5 \times 5 = 125$. So using quantifiers can pay off even when they are not strictly necessary. ■

Problem 4. **(a)** Explain why

$$(\forall z.\ P(z, z)) \longrightarrow \forall x \exists y.\ P(x, y) \tag{1}$$

is valid.

Solution. *Proof.* Assume

$$\forall z.\ P(z, z) \tag{2}$$

is true for some domain and interpretation of the predicate P. We want to show that

$$\forall x \exists y.\ P(x, y) \tag{3}$$

also holds.

So let c be an element of the domain. Then $P(c, c)$ holds by assumption (2). So there is a y, namely $y = c$ such that $P(c, y)$ holds. That is, $\exists y.\ P(c, y)$ is true. But c could have been any element in the domain, so (by *Universal Generalization*), we conclude that (3) holds. □

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(b) Describe a counter-model demonstrating that

$$(\forall x \exists y.\ P(x, y)) \longrightarrow \forall z.\ P(z, z)$$

is not valid.

Solution. Let $P(x, y)$ mean $x \neq y$. Then the conclusion $\forall z.\ z \neq z$ is always false, but in any domain with two or more elements, the hypothesis is true. ■